# Conformal self-dual fields

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**Abstract**
Conformal self-dual fields in flat spacetime of even dimension greater than
or equal to 4 are studied. Ordinary-derivative formulation of such fields
is developed. Gauge invariant Lagrangian with conventional kinetic terms
and corresponding gauge transformations are obtained. Gauge symmetries
are realized by involving the Stueckelberg fields. The realization of global
conformal symmetries is obtained. The light-cone gauge Lagrangian is found.
Also, we demonstrate the use of the light-cone gauge for counting the on-shell
degrees of freedom of the conformal self-dual fields.

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## 1. Introduction

In Poincaré and conformal supergravity theories, the self-duality manifests itself in different
ways. In Poincaré supergravity theories, some of the antisymmetric tensor fields are not self-
dual, while their field strengths are self-dual (see, e.g., [1]).[1] In contrast to this, in conformal
supergravity theories, some of the antisymmetric tensor fields are self-dual themselves, while
their field strengths are not self-dual (see, e.g., [6]).

   In Poincaré supergravity theories, the antisymmetric tensor fields are realized as gauge
fields, while in the standard approach to conformal supergravity theories there are no
gauge symmetries related to the self-dual antisymmetric tensor fields. Note also that the
antisymmetric tensor fields of Poincaré supergravity theories describe ghost-free dynamics,
while the ones of conformal supergravity theories contain ghost degrees of freedom.

   In this paper, we discuss the self-dual antisymmetric tensor fields of conformal
supergravity theories (which are well defined only in $d = 4, 6$) and their counterparts in

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[1] It is the self-duality of the field strength that leads to the problem with Lorentz invariant action for the gauge
antisymmetric tensor field [2] without the use of auxiliary fields. The study of the Lorentz covariant formulations
involving auxiliary fields may be found in [3, 4]. An interesting discussion of self-dual fields in $d = 6, 10$ may be
found in [5].

spacetimes of arbitrary even dimensions. It is these self-dual antisymmetric tensor fields that will be referred to as conformal self-dual fields, or shortly as self-dual fields in this paper.

The standard formulation of the self-dual fields involves exotic kinetic terms[2]. These exotic kinetic terms can be re-expressed in terms of kinetic terms involving the standard Dalambertian operator but this leads to higher derivatives (see e.g. [6]). Also, as was mentioned above, in the standard approach, there are no gauge symmetries associated with such self-dual fields.

The purpose of this paper is to develop an ordinary (not higher-) derivative, gauge invariant and Lagrangian formulation for the self-dual fields[3]. In this paper, we discuss free self-dual fields in the spacetime of even dimension $d \geqslant 4$. Our approach to the self-dual fields can be summarized as follows.

 (i) We introduce additional field degrees of freedom (D.o.F.), i.e. we extend the space of fields entering the standard formulation of self-dual fields. These additional field D.o.F. are supplemented by appropriate gauge symmetries[4]. We note that these additional field D.o.F. are similar to the ones used in the gauge invariant formulation of massive fields. Sometimes, such additional field D.o.F. are referred to as Stueckelberg fields.

 (ii) Our Lagrangian for the free self-dual fields does not involve higher than second-order terms in derivatives. Two-derivative contributions to the Lagrangian take the form of the standard kinetic terms of the antisymmetric tensor fields. The Lagrangian is invariant under gauge transformations and global conformal algebra transformations.

(iii) Gauge transformations of the free self-dual fields do not involve higher than first-order terms in derivatives. One-derivative contributions to the gauge transformations take the form of the standard gauge transformations of the antisymmetric tensor fields.

(iv) The gauge symmetries of our Lagrangian make it possible to match our approach with the standard one, i.e. by an appropriate gauge fixing of the Stueckelberg fields and by solving some constraints, we obtain the standard formulation of the self-dual fields. This implies that our approach retain on-shell D.o.F. of the standard theory of self-dual fields, i.e. on-shell, our approach is equivalent to the standard one.

As is well known, the Stueckelberg approach turned out to be successful for the study of theories involving massive fields. That is to say that all covariant formulations of string theories are realized by using Stueckelberg gauge symmetries. The self-dual fields enter the field content of conformal supergravity theories. Therefore, we expect that the use of the Stueckelberg fields for studying the self-dual fields might be useful for developing new interesting formulations of the conformal supergravity theories.

The rest of the paper is organized as follows.

In section 2, we summarize the notation and review the standard approach to the self-dual fields.

In section 3, we start with the example of a self-dual field propagating in $4d$ Minkowski space. For this field, we obtain the ordinary-derivative gauge invariant Lagrangian. We find the realization of the conformal $so(4, 2)$ algebra symmetries on the space of gauge fields and on the space of field strengths. Also we obtain the light-cone gauge Lagrangian and demonstrate that the number of on-shell D.o.F. of our approach coincides with the one in the

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[2] For instance, the self-dual field $T^{ab}$ of $\mathcal{N} = 4$, $4d$ conformal supergravity is described by the Lagrangian $\mathcal{L} = \partial^a \bar{T}^{ab} \partial^c T^{cb}$.

[3] Making comparison with various approaches to massive fields, one can say that the standard approach to the self-dual fields is a counterpart of the Pauli–Fierz approach to the massive fields, while our approach to the self-dual fields is a counterpart of the Stueckelberg approach to the massive fields.

[4] To realize those additional gauge symmetries, we adopt the approach of [7, 8] which turns out to be the most useful for our purposes.

standard approach to the self-dual field. We discuss the decomposition of those on-shell D.o.F into irreps of the $so(2)$algebra.

In section 4, we generalize results obtained in section 3 to the case of self-dual fields propagating in Minkowski space of arbitrary dimension.

In section 5, we represent our results in sections 3 and 4 by using the realization of field degrees of freedom in terms of generating functions. The generating functions are constructed out of the self-dual gauge fields and some oscillators. The use of the generating functions simplifies considerable study of the self-dual fields. Therefore, we believe that the result in section 5 might be helpful in future studies of the self-dual fields.

Section 6 suggests directions for future research.

We collect various technical details in the appendices. In appendix A, we discuss details of the derivation of the ordinary-derivative gauge invariant Lagrangian. In appendix B, we present details of the derivation of the conformal algebra transformations of gauge fields. In appendix C, we discuss some details of the derivation of the light-cone gauge Lagrangian. In appendix D, we collect some useful formulas involving the Levi-Civita symbol.

## 2. Preliminaries

### 2.1. Notation

Throughout the paper, the dimension of a flat spacetime, which we denote by $d$, is restricted to be an even integer, $d = 2\nu$. Coordinates in the flat spacetime are denoted by $x^a$, while $\partial_a$ stands for the derivative with respect to $x^a$, $\partial_a \equiv \partial/\partial x^a$. Vector indices of the Lorentz algebra $so(d-1, 1)$ take the values $a, b, c, e = 0, 1, \ldots, d-1$. To simplify our expressions, we drop $\eta_{ab}$ in scalar products, i.e. we use $X^a Y^a \equiv \eta_{ab} X^a Y^b$. The notation $\epsilon^{a_1 \ldots a_\nu b_1 \ldots b_\nu}$ stands for the Levi-Civita symbol. We assume the normalization $\epsilon^{01 \ldots d-1} = 1$.

To avoid complicated tensor expressions, we use a set of the creation operators $\alpha^a$, $\zeta$, $\upsilon^\oplus$, $\upsilon^\ominus$, and the respective set of annihilation operators $\bar\alpha^a$, $\bar\zeta$, $\bar\upsilon^\ominus$, $\bar\upsilon^\oplus$:

$$\bar\alpha^a|0\rangle = 0, \qquad \bar\zeta|0\rangle = 0, \qquad \bar\upsilon^\oplus|0\rangle = 0, \qquad \bar\upsilon^\ominus|0\rangle = 0. \tag{2.1}$$

These operators satisfy the following (anti)commutation relations:

$$\{\bar\alpha^a, \alpha^b\} = \eta^{ab}, \qquad \{\bar\zeta, \zeta\} = 1, \tag{2.2}$$

$$[\bar\upsilon^\oplus, \upsilon^\ominus] = 1, \qquad [\bar\upsilon^\ominus, \upsilon^\oplus] = 1, \tag{2.3}$$

and will often be referred to as oscillators in what follows[5]. The oscillators $\alpha^a$, $\bar\alpha^a$ and $\zeta$, $\bar\zeta$, $\upsilon^\oplus$, $\upsilon^\ominus$, $\bar\upsilon^\oplus$, $\bar\upsilon^\ominus$ transform in the respective vector and scalar representations of the $so(d-1, 1)$ Lorentz algebra and satisfy the following Hermitian conjugation rules:

$$\alpha^{a\dagger} = \bar\alpha^a, \qquad \zeta^\dagger = \bar\zeta, \qquad \upsilon^{\oplus\dagger} = \bar\upsilon^\oplus, \qquad \upsilon^{\ominus\dagger} = \bar\upsilon^\ominus. \tag{2.4}$$

Throughout this paper, we use operators constructed out of the oscillators and derivatives

$$\Box = \partial^a \partial^a, \qquad \alpha\partial = \alpha^a \partial^a, \qquad \bar\alpha\partial = \bar\alpha^a \partial^a, \tag{2.5}$$

$$N_\alpha \equiv \alpha^a \bar\alpha^a, \tag{2.6}$$

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[5] We use oscillator formulation [9–14] to handle the many indices appearing for tensor fields. It can also be reformulated as an algebra acting on the symmetric-spinor bundle on the manifold $M$ [15]. Note that the scalar oscillators $\zeta$, $\bar\zeta$ arise naturally by a dimensional reduction from flat space. It is natural to expect that the 'conformal' oscillators $\upsilon^\oplus$, $\upsilon^\ominus$, $\bar\upsilon^\oplus$, $\bar\upsilon^\ominus$ also allow certain interpretation via dimensional reduction. Interesting recent discussion of dimensional reduction may be found in [16].

$$N_\zeta \equiv \zeta\bar{\zeta}, \tag{2.7}$$

$$N_{\upsilon\oplus} \equiv \upsilon^\oplus\bar{\upsilon}^\ominus, \tag{2.8}$$

$$N_{\upsilon\ominus} \equiv \upsilon^\ominus\bar{\upsilon}^\oplus, \tag{2.9}$$

$$N_\upsilon \equiv N_{\upsilon\oplus} + N_{\upsilon\ominus}. \tag{2.10}$$

## 2.2. Global conformal symmetries

In the spacetime of dimension $d$, the conformal algebra $so(d, 2)$ referred to the basis of Lorentz algebra $so(d - 1, 1)$ consists of translation generators $P^a$, conformal boost generators $K^a$, dilatation generator $D$ and generators $so(d - 1, 1)$ Lorentz algebra $J^{ab}$. We assume the following normalization for commutators of the conformal algebra:

$$[D, P^a] = -P^a, \qquad [P^a, J^{bc}] = \eta^{ab}P^c - \eta^{ac}P^b, \tag{2.11}$$

$$[D, K^a] = K^a, \qquad [K^a, J^{bc}] = \eta^{ab}K^c - \eta^{ac}K^b, \tag{2.12}$$

$$[P^a, K^b] = \eta^{ab}D - J^{ab}, \tag{2.13}$$

$$[J^{ab}, J^{ce}] = \eta^{bc}J^{ae} + 3 \text{ terms.} \tag{2.14}$$

Let $|\phi\rangle$ denote the field propagating in the flat spacetime of dimension $d \geqslant 4$. Let the Lagrangian for the free field $|\phi\rangle$ be conformal invariant. This implies that the Lagrangian is invariant with respect to the transformation (invariance of the Lagrangian is assumed to be up to the total derivative)

$$\delta_{\hat{G}}|\phi\rangle = \hat{G}|\phi\rangle, \tag{2.15}$$

where a realization of the conformal algebra generators $\hat{G}$ in terms of differential operators takes the form

$$P^a = \partial^a, \tag{2.16}$$

$$J^{ab} = x^a\partial^b - x^b\partial^a + M^{ab}, \tag{2.17}$$

$$D = x\partial + \Delta, \tag{2.18}$$

$$K^a = K^a_{\Delta,M} + R^a, \tag{2.19}$$

$$K^a_{\Delta,M} \equiv -\tfrac{1}{2}x^2\partial^a + x^a D + M^{ab}x^b, \tag{2.20}$$

$$x\partial \equiv x^a\partial^a, \qquad x^2 = x^a x^a. \tag{2.21}$$

In (2.17)–(2.19), $\Delta$ is the operator of conformal dimension, $M^{ab}$ is the spin operator of the Lorentz algebra,

$$[M^{ab}, M^{ce}] = \eta^{bc}M^{ae} + 3 \text{ terms,} \tag{2.22}$$

and $R^a$ is the operator depending on the derivative $\partial^a$ and not depending on the spacetime coordinates $x^a$, $[P^a, R^b] = 0$. The spin operator $M^{ab}$ is well known for arbitrary tensor fields of the Lorentz algebra. In the standard formulation of the self-dual fields, the operator $R^a$ is equal to zero, while in the ordinary-derivative approach, we develop in this paper, the operator $R^a$ is non-trivial. This implies that, in the framework of the ordinary-derivative approach, the complete description of the self-dual fields requires finding not only the gauge invariant Lagrangian but also the operator $R^a$.

An explicit representation for the action of the operator $K^a_{\Delta,M}$ (2.20) is easily obtained from the relations given above. Let $\Lambda^{a_1...a_n}$ be the rank-$n$ antisymmetric tensor field of the Lorentz algebra $so(d-1,1)$, while $\Delta(\Lambda)$ is a conformal dimension of this tensor field. Relation (2.19) implies that the conformal boost transformations of $\Lambda^{a_1...a_n}$ can be presented as

$$\delta_{K^a}\Lambda^{a_1...a_n} = \delta_{K^a_{\Delta,M}}\Lambda^{a_1...a_n} + \delta_{R^a}\Lambda^{a_1...a_n}, \tag{2.23}$$

$$\delta_{K^a_{\Delta,M}}\Lambda^{a_1...a_n} = K^a_{\Delta(\Lambda)}\Lambda^{a_1...a_n} + \sum_{k=1}^{n} M^{aa_k c}\Lambda^{a_1...a_{k-1}ca_{k+1}...a_n}, \tag{2.24}$$

$$K^a_\Delta \equiv -\frac{1}{2}x^2\partial^a + x^a(x\partial + \Delta), \tag{2.25}$$

$$M^{abc} \equiv \eta^{ab}x^c - \eta^{ac}x^b. \tag{2.26}$$

Thus, all that remains is to find explicit representation for the operator $R^a$. This is what we are doing, among other things, in this paper.

### 2.3. Standard approach to self-dual fields

We begin with a brief review of the standard approach to the conformal self-dual fields. In this section we recall the main facts of conformal field theory about these fields.

Consider the totally antisymmetric rank-$\nu$ tensor field $T^{a_1...a_\nu}$, $\nu \equiv \frac{d}{2}$, of the Lorentz algebra $so(d-1,1)$, where the dimension of spacetime $d$ is an even integer. In the framework of a standard approach, the field $T^{a_1...a_\nu}$ is referred to as a conformal self-dual field if it meets the following requirements.

(a) The field $T^{a_1...a_\nu}$ satisfies the self-duality constraint

$$T^{a_1...a_\nu} = \frac{\tau}{\nu!}\epsilon^{a_1...a_\nu b_1...b_\nu}T^{b_1...b_\nu}, \tag{2.27}$$

$$\tau = \begin{cases} \pm i & \text{for} \quad d = 4k; \\ \pm 1 & \text{for} \quad d = 4k+2, \end{cases} \tag{2.28}$$

where $\epsilon^{a_1...a_\nu b_1...b_\nu}$ is the Levi-Civita symbol. For flexibility, we do not fix the sign of $\tau$. Constraint (2.27) implies that $T^{a_1...a_\nu}$ is *complex-valued* when $d = 4k$. In $d = 4k+2$, the field $T^{a_1...a_\nu}$ is considered to be *real-valued*.

(b) Dynamics of the field $T^{a_1...a_\nu}$ is described by the Lagrangian

$$\mathcal{L}_{st} = \frac{1}{(\nu-1)!}\partial^a\bar{T}^{aa_2...a_\nu}\partial^b T^{ba_2...a_\nu}, \qquad \text{for} \quad d = 4k, \tag{2.29}$$

$$\mathcal{L}_{st} = \frac{1}{(\nu-1)!}\partial^a T^{aa_2...a_\nu}\partial^b T^{ba_2...a_\nu}, \qquad \text{for} \quad d = 4k+2, \tag{2.30}$$

where $\bar{T}$ in (2.29) stands for the complex conjugate of $T$.

We now note the following.

(i) Requiring the Lagrangian to be invariant under the dilatation transformation, we obtain the conformal dimension of the field $T^{a_1...a_\nu}$,

$$\Delta(T^{a_1...a_\nu}) = \frac{d-2}{2}, \tag{2.31}$$

which is referred to as *the canonical conformal dimension of the conformal self-dual field*.

(ii) The operator $R^a$ of the field $T^{a_1 \dots a_v}$ is equal to zero.

(iii) The simplest case of the self-dual field, which is an antisymmetric complex-valued rank-2 tensor field $T^{ab}$, corresponds to $d = 4$ with the following self-duality constraint (see (2.27)):

$$T^{ab} = \frac{\tau}{2} \epsilon^{abce} T^{ce}. \tag{2.32}$$

The Lagrangian for the field $T^{ab}$ can be read from (2.29):

$$\mathcal{L}_{\text{st}} = \partial^a \bar{T}^{ac} \partial^b T^{bc}. \tag{2.33}$$

The field $T^{ab}$ appears in the field content of $\mathcal{N} = 4$, $4d$ conformal supergravity.

## 3. Ordinary-derivative approach to the self-dual field for $d = 4$

As a warm up, let us start with the simplest case of the self-dual fields. Consider the self-dual field $T^{ab}$ propagating in $4d$ flat space. In the framework of the ordinary-derivative approach, a dynamical system whose on-shell equivalent to the self-dual field $T^{ab}$ with the Lagrangian (2.33) involves two vector fields $\phi^a_{-1}$ and $\phi^a_1$, one scalar field $\phi_0$ and one self-dual rank-2 tensor field $t^{ab}$. In other words, we use the following field content:

$$\phi^a_{-1}, \qquad \phi^a_1, \qquad \phi_0, \qquad t^{ab}, \tag{3.1}$$

$$t^{ab} = \frac{\tau}{2} \epsilon^{abce} t^{ce}, \tag{3.2}$$

$\tau = \pm \mathrm{i}$. All fields in (3.1) are *complex-valued*. Conformal dimensions of these fields are given by

$$\Delta\left(\phi^a_{-1}\right) = 0, \qquad \Delta\left(\phi^a_1\right) = 2, \qquad \Delta(\phi_0) = 1, \qquad \Delta(t^{ab}) = 1. \tag{3.3}$$

We note that the subscript $k'$ in $\phi_{k'}$ implies that the conformal dimension of the field $\phi_{k'}$ is equal to $1 + k'$.

The ordinary-derivative action and Lagrangian we found take the form

$$S = \int \mathrm{d}^4 x \mathcal{L}, \tag{3.4}$$

$$\mathcal{L} = -\frac{1}{2} F^{ab}(\bar{\phi}_{-1}) F^{ab}(\phi_1) - \frac{1}{2} F^{ab}(\bar{\phi}_1) F^{ab}(\phi_{-1})$$
$$-\frac{1}{2} \bar{t}^{ab} F^{ab}(\phi_1) - \frac{1}{2} F^{ab}(\bar{\phi}_1) t^{ab} - \left(\bar{\phi}^a_1 + \partial^a \bar{\phi}_0\right)\left(\phi^a_1 + \partial^a \phi_0\right), \tag{3.5}$$

where $\bar{\phi}_{k'}$ and $\bar{t}^{ab}$ are the respective complex conjugates of $\phi_{k'}$ and $t^{ab}$, while $F^{ab}(\phi)$ stands for the field strength defined as

$$F^{ab}(\phi) \equiv \partial^a \phi^b - \partial^b \phi^a. \tag{3.6}$$

Details of the derivation of Lagrangian (3.5) may be found in appendix A.

A few remarks are in order.

(i) Two-derivative contributions to Lagrangian (3.5) are the standard kinetic terms for the vector fields $\phi^a_{-1}$, $\phi^a_1$ and the standard Klein–Gordon kinetic term for the scalar field $\phi_0$.

(ii) In addition to the two-derivative contributions, the Lagrangian involves one-derivative contributions and derivative-independent mass-like contributions. The appearance of the one-derivative and derivative-independent contributions to the Lagrangian is a characteristic feature of the ordinary-derivative approach.

(iii) The self-dual field $t^{ab}$ plays the role of a Lagrangian multiplier. Equations of motion for $t^{ab}$ together with the self-duality constraint for $t^{ab}$ (3.2) tell us that *on-shell* the field strength $F^{ab}(\phi_1)$ is self-dual:

$$F^{ab}(\phi_1) = \frac{\tau}{2}\epsilon^{abce}F^{ce}(\phi_1). \tag{3.7}$$

### 3.1. Gauge transformations

To discuss gauge symmetries of Lagrangian (3.5), we introduce the following gauge transformation parameters:

$$\xi_{-2}, \qquad \xi_0, \qquad \lambda^a. \tag{3.8}$$

All these gauge transformation parameters are complex-valued. Conformal dimensions of the gauge transformation parameters are given by

$$\Delta(\xi_{-2}) = -1, \qquad \Delta(\xi_0) = 1, \qquad \Delta(\lambda^a) = 0. \tag{3.9}$$

We find that Lagrangian (3.5) is invariant under the gauge transformations

$$\delta\phi_1^a = \partial^a \xi_0, \tag{3.10}$$

$$\delta\phi_{-1}^a = \partial^a \xi_{-2} - \lambda^a, \tag{3.11}$$

$$\delta\phi_0 = -\xi_0, \tag{3.12}$$

$$\delta t^{ab} = F^{ab}(\lambda) + \frac{\tau}{2}\epsilon^{abce}F^{ce}(\lambda), \tag{3.13}$$

where the field strength $F^{ab}(\lambda)$ for the gauge transformation parameter $\lambda^a$ is defined as in (3.6).

From (3.11), (3.12), we see that the vector and scalar fields, $\phi_{-1}^a$, $\phi_0$, transform as Stueckelberg fields, i.e. these fields can be gauged away via Stueckelberg gauge fixing, $\phi_{-1}^a = 0$, $\phi_0 = 0$. If we gauge away these fields, and exclude the vector field $\phi_1^a$ via equations of motion, then our Lagrangian reduces to the Lagrangian of the standard approach (2.33). Note also that, in the Stueckelberg gauge, the field $t^{ab}$ is identified with the generic self-dual field $T^{ab}$. Thus, our approach is equivalent to the standard one.

### 3.2. Realization of conformal algebra symmetries

To complete the ordinary-derivative description of the conformal self-dual field, we should provide the realization of the conformal algebra symmetries on a space of fields (3.1). The Poincaré algebra symmetries are realized on fields (3.1) in a standard way. The realization of dilatation symmetry is given by (2.18), where conformal dimensions of fields (3.1) are given in (3.3). A general realization of conformal boost symmetries on arbitrary tensor fields is given in (2.23). According to (2.23), we should find realizations of the operators $K_{\Delta,M}^a$ and $R^a$ on a space of fields (3.1). The realization of the former operator is obtained by adopting the general formula (2.24) for gauge fields (3.1):

$$\delta_{K_{\Delta,M}^a}\phi_{k'}^b = K_{\Delta(\phi_{k'})}^a\phi_{k'}^b + M^{abc}\phi_{k'}^c, \qquad k' = \pm 1, \tag{3.14}$$

$$\delta_{K_{\Delta,M}^a}\phi_0 = K_{\Delta(\phi_0)}^a\phi_0, \tag{3.15}$$

$$\delta_{K_{\Delta,M}^a}t^{a_1 a_2} = K_{\Delta(t)}^a t^{a_1 a_2} + M^{aa_1 c}t^{ca_2} + M^{aa_2 c}t^{a_1 c}, \tag{3.16}$$

where the operator $K_\Delta^a$ and conformal dimensions are defined in (2.25) and (3.3), respectively.

The real difficulty is to find the operator $R^a$. The realization of the operator $R^a$ on a space of gauge fields (3.1) that we found is given by

$$\delta_{R^a}\phi_1^b = -t^{ab} - \eta^{ab}\phi_0 - \partial^a\phi_{-1}^b - \frac{\tau}{2}\epsilon^{abce}F^{ce}(\phi_{-1}),\tag{3.17}$$

$$\delta_{R^a}\phi_{-1}^b = 0\,,\tag{3.18}$$

$$\delta_{R^a}\phi_0 = \phi_{-1}^a\,,\tag{3.19}$$

$$\delta_{R^a}t^{a_1a_2} = \eta^{aa_1}\phi_{-1}^{a_2} - \eta^{aa_2}\phi_{-1}^{a_1} + \tau\epsilon^{a_1a_2ab}\phi_{-1}^b.\tag{3.20}$$

Using (3.14)–(3.20) and general formula (2.23),

$$\delta_{K^a} = \delta_{K_{\Delta,M}^a} + \delta_{R^a}\,,\tag{3.21}$$

gives the conformal boost transformations of the gauge fields.

From (3.17)–(3.20), we see that the operator $R^a$ maps the gauge field with a conformal dimension $\Delta$ into the ones having a conformal dimension less than $\Delta$. This is to say that the realization of the operator $R^a$ given in (3.17)–(3.20) can schematically be represented as

$$\phi_1 \xrightarrow{R} t \oplus \phi_0 \oplus \partial\phi_{-1},\tag{3.22}$$

$$t \xrightarrow{R} \phi_{-1}, \qquad \phi_0 \xrightarrow{R} \phi_{-1}, \qquad \phi_{-1} \xrightarrow{R} 0.\tag{3.23}$$

Details of the derivation of the operator $R^a$ may be found in appendix B.[6] As a side of remark we note that, having introduced the field content and the Lagrangian, the operator $R^a$ is fixed uniquely by requiring that

(i) the operator $R^a$ should not involve higher than first-order terms in the derivative;
(ii) the Lagrangian should be invariant under the conformal algebra transformations.

As usual, the conformal algebra transformations of gauge fields (3.1) are defined up to gauge transformations. Alternatively, the conformal algebra symmetries can be realized on a space of field strengths. We now discuss field strengths for gauge fields (3.1) and the corresponding conformal transformations of the field strengths.

### 3.3. Realization of conformal algebra symmetries on the space of field strengths

We introduce the following field strengths which are constructed out of gauge fields (3.1):

$$F^{\ominus ab} = T^{ab},\tag{3.24}$$

$$F^{abc} = F^{abc}(T)\,,\tag{3.25}$$

$$F^{\oplus\ominus a} = \phi_1^a + \partial^a\phi_0,\tag{3.26}$$

$$F^{\oplus ab} = F^{ab}(\phi_1)\,,\tag{3.27}$$

where $F^{ab}(\phi_1)$ is defined as in (3.6), while $T^{ab}$ and $F^{abc}(T)$ are defined by the respective relations:

$$T^{ab} = t^{ab} + F^{ab}(\phi_{-1}) + \frac{\tau}{2}\epsilon^{abce}F^{ce}(\phi_{-1}),\tag{3.28}$$

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[6] In appendix B, the operator $R^a$ is obtained by using the realization of field D.o.F. in terms of generating functions. The discussion of the generating functions may be found in section 5. Therefore, before reading appendix B, the reader should consult section 5.

$$F^{abc}(T) = \partial^a T^{bc} + \partial^b T^{ca} + \partial^c T^{ab} \,. \tag{3.29}$$

One can make sure that field strengths (3.24)–(3.27) are invariant under gauge transformations (3.10)–(3.13). Conformal dimensions of the field strengths can easily be read from relations (3.3) and (3.24)–(3.27):

$$\Delta(F^{\ominus ab}) = 1, \qquad \Delta(F^{abc}) = 2, \qquad \Delta(F^{\oplus\ominus a}) = 2, \qquad \Delta(F^{\oplus ab}) = 3. \tag{3.30}$$

Poincaré algebra symmetries are realized on a space of the field strengths in a usual way. The realization of dilatation symmetry is given by (2.18), where conformal dimensions of the field strengths are given in (3.30). All that remains is to find conformal boost transformations of the field strengths. Making use of the conformal boost transformations of the gauge fields defined by relations (3.14)–(3.20), we find the corresponding conformal boost transformations of the field strengths. Those conformal boost transformations of the field strengths can be represented as in (2.23), (2.24), where we substitute, in place of the tensor fields $\Lambda^{a_1 \dots a_n}$, the fields strengths with the following realization of the operator $R^a$:

$$\delta_{R^a} F^{\ominus bc} = 0 \,, \tag{3.31}$$

$$\delta_{R^a} F^{bce} = -\eta^{ab} F^{\ominus ce} - \eta^{ac} F^{\ominus eb} - \eta^{ae} F^{\ominus bc} \,, \tag{3.32}$$

$$\delta_{R^a} F^{\oplus\ominus b} = -F^{\ominus ab} \,, \tag{3.33}$$

$$\delta_{R^a} F^{\oplus bc} = \eta^{ab} F^{\oplus\ominus c} - \eta^{ac} F^{\oplus\ominus b} + F^{abc} - \partial^a F^{\ominus bc}. \tag{3.34}$$

From (3.31)–(3.34), we see that the operator $R^a$ maps the field strength with a conformal dimension $\Delta$ into the ones having a conformal dimension less than $\Delta$. In other words, the realization of the operator $R^a$ given in (3.31)–(3.34) can schematically be represented as

$$F^{\oplus} \xrightarrow{R} F^{\oplus\ominus} \oplus F \oplus \partial F^{\ominus}, \tag{3.35}$$

$$F \xrightarrow{R} F^{\ominus}, \qquad F^{\oplus\ominus} \xrightarrow{R} F^{\ominus}, \qquad F^{\ominus} \xrightarrow{R} 0. \tag{3.36}$$

### 3.4. On-shell degrees of freedom and light-cone gauge Lagrangian

In order to discuss on-shell D.o.F. of the conformal self-dual field, we use a nomenclature of the $so(d-2)$ algebra which is $so(2)$ when $d = 4$. Namely, we decompose the on-shell D.o.F. into irreps of the $so(2)$ algebra. One can prove that the on-shell D.o.F. of the self-dual field are described by two $so(2)$ algebra self-dual complex-valued vector fields $\phi^i_{-1}$, $\phi^i_1$ and one complex-valued scalar fields $\phi_0$:

$$\phi^i_{-1}, \qquad \phi^i_1, \qquad \phi_0, \tag{3.37}$$

where vector indices of the $so(2)$ algebra take values $i, j = 1, 2$. The vector fields satisfy the $so(2)$ self-duality constraint:

$$\phi^i_{-1} = \tau^{-1} \epsilon^{ij} \phi^j_{-1}, \qquad \phi^i_1 = \tau^{-1} \epsilon^{ij} \phi^j_1, \tag{3.38}$$

where $\epsilon^{ij}$ is the Levi-Civita symbol normalized as $\epsilon^{12} = 1$.

Using the light-cone gauge frame, one can make sure that the gauge invariant Lagrangian (3.5) leads to the following light-cone gauge Lagrangian for fields (3.37):

$$\mathcal{L}_{\text{l.c.}} = \bar{\phi}^i_1 \Box \phi^i_{-1} + \bar{\phi}^i_{-1} \Box \phi^i_1 + \bar{\phi}_0 \Box \phi_0 - \bar{\phi}^i_1 \phi^i_1. \tag{3.39}$$

Details of the derivation of the on-shell D.o.F. and the light-cone gauge Lagrangian may be found in appendix C.

From (3.37), (3.38), we see that the number of real-valued on-shell D.o.F. is equal to 6. This result agrees with the one found in [6]. Note however that we not only find the number of the on-shell D.o.F. but also provide the decomposition of those on-shell D.o.F. into irreps of the $so(2)$ algebra.

## 4. Ordinary-derivative approach to the self-dual field for arbitrary $d = 2\nu$

We now develop the ordinary-derivative approach to the conformal self-dual field propagating in flat spacetime of an arbitrary even dimension $d = 2\nu$. To discuss the ordinary-derivative approach to the self-dual field, we use the following field content:

$$\phi_{-1}^{a_1...a_{\nu-1}}, \qquad \phi_1^{a_1...a_{\nu-1}}, \qquad \phi_0^{a_1...a_{\nu-2}}, \qquad t^{a_1...a_\nu}, \tag{4.1}$$

where the field $t^{a_1...a_\nu}$ satisfies the self-duality constraint

$$t^{a_1...a_\nu} = \frac{\tau}{\nu!} \epsilon^{a_1...a_\nu b_1...b_\nu} t^{b_1...b_\nu}, \tag{4.2}$$

and $\tau$ is defined in (2.28). We note that

(i) fields in (4.1) are antisymmetric tensor fields of the Lorentz algebra $so(d-1,1)$;
(ii) fields in (4.1) are complex-valued when $d = 4k$ and real-valued when $d = 4k+2$;
(iii) conformal dimensions of fields in (4.1) are given by

$$\Delta\big(\phi_{-1}^{a_1...a_{\nu-1}}\big) = \frac{d-4}{2}, \qquad \Delta\big(\phi_1^{a_1...a_{\nu-1}}\big) = \frac{d}{2}, \qquad \Delta\big(\phi_0^{a_1...a_{\nu-2}}\big) = \frac{d-2}{2},$$

$$\Delta(t^{a_1...a_\nu}) = \frac{d-2}{2}. \tag{4.3}$$

We note that the subscript $k'$ in $\phi_{k'}$ implies that the conformal dimension of the field $\phi_{k'}$ is equal to $\frac{d-2}{2} + k'$.

An ordinary-derivative action that we found is given by

$$S = \int d^d x \, \mathcal{L}, \tag{4.4}$$

where the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{\nu!} F^{a_1...a_\nu}(\bar{\phi}_{-1}) F^{a_1...a_\nu}(\phi_1) - \frac{1}{\nu!} F^{a_1...a_\nu}(\bar{\phi}_1) F^{a_1...a_\nu}(\phi_{-1})$$

$$- \frac{1}{\nu!} \bar{t}^{a_1...a_\nu} F^{a_1...a_\nu}(\phi_1) - \frac{1}{\nu!} t^{a_1...a_\nu} F^{a_1...a_\nu}(\bar{\phi}_1)$$

$$- \frac{1}{(\nu-1)!} \big(\bar{\phi}_1^{a_1...a_{\nu-1}} + F^{a_1...a_{\nu-1}}(\bar{\phi}_0)\big)\big(\phi_1^{a_1...a_{\nu-1}} + F^{a_1...a_{\nu-1}}(\phi_0)\big), \tag{4.5}$$

when $d = 4k$, while for $d = 4k+2$, the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{\nu!} F^{a_1...a_\nu}(\phi_{-1}) F^{a_1...a_\nu}(\phi_1) - \frac{1}{\nu!} t^{a_1...a_\nu} F^{a_1...a_\nu}(\phi_1)$$

$$- \frac{1}{2(\nu-1)!} \big(\phi_1^{a_1...a_{\nu-1}} + F^{a_1...a_{\nu-1}}(\phi_0)\big)\big(\phi_1^{a_1...a_{\nu-1}} + F^{a_1...a_{\nu-1}}(\phi_0)\big), \tag{4.6}$$

where the field strengths are defined as

$$F^{a_1...a_n}(\phi) = n \partial^{[a_1} \phi^{a_2...a_n]} \tag{4.7}$$

and the antisymmetrization of the tensor indices is normalized as $[a_1 \ldots a_n] = \frac{1}{n!}(a_1 \ldots a_n \pm (n!-1)$ terms).

We note that

(i) two-derivative contributions to Lagrangians (4.5), (4.6) take the form of standard second-order kinetic terms for antisymmetric tensor fields. Besides the two-derivative contributions, the Lagrangians involve one-derivative contributions and derivative-independent mass-like contributions.

(ii) Equations of motion for $t^{a_1 \ldots a_v}$ and the self-duality constraint for $t^{a_1 \ldots a_v}$ (4.2) imply that *on-shell* the field strength $F^{a_1 \ldots a_v}(\phi_1)$ satisfies the self-duality constraint:

$$F^{a_1 \ldots a_v}(\phi_1) = \frac{\tau}{v!} \epsilon^{a_1 \ldots a_v b_1 \ldots b_v} F^{b_1 \ldots b_v}(\phi_1). \tag{4.8}$$

### 4.1. Gauge transformations

We now discuss gauge symmetries of Lagrangians (4.5), (4.6). To this end, we introduce the following gauge transformation parameters:

$$\xi_{-2}^{a_1 \ldots a_{v-2}}, \qquad \xi_0^{a_1 \ldots a_{v-2}}, \qquad \xi_{-1}^{a_1 \ldots a_{v-3}}, \qquad \lambda^{a_1 \ldots a_{v-1}}. \tag{4.9}$$

We note that

(i) gauge transformation parameters (4.9) are antisymmetric tensor fields of the Lorentz algebra $so(d-1, 1)$;

(ii) gauge transformation parameters (4.9) are complex-valued when $d = 4k$ and real-valued when $d = 4k + 2$;

(iii) conformal dimensions of gauge transformation parameters (4.9) are given by

$$\Delta\big(\xi_{-1}^{a_1 \ldots a_{v-2}}\big) = \frac{d-6}{2}, \qquad \Delta\big(\xi_0^{a_1 \ldots a_{v-2}}\big) = \frac{d-2}{2}, \qquad \Delta\big(\xi_{-1}^{a_1 \ldots a_{v-3}}\big) = \frac{d-4}{2},$$

$$\Delta(\lambda^{a_1 \ldots a_{v-1}}) = \frac{d-4}{2}. \tag{4.10}$$

Gauge transformations that we found take the form

$$\delta\phi_1^{a_1 \ldots a_{v-1}} = F^{a_1 \ldots a_{v-1}}(\xi_0), \tag{4.11}$$

$$\delta\phi_{-1}^{a_1 \ldots a_{v-1}} = F^{a_1 \ldots a_{v-1}}(\xi_{-2}) - \lambda^{a_1 \ldots a_{v-1}}, \tag{4.12}$$

$$\delta\phi_0^{a_1 \ldots a_{v-2}} = F^{a_1 \ldots a_{v-2}}(\xi_{-1}) - \xi_0^{a_1 \ldots a_{v-2}}, \tag{4.13}$$

$$\delta t^{a_1 \ldots a_v} = F^{a_1 \ldots a_v}(\lambda) + \frac{\tau}{v!} \epsilon^{a_1 \ldots a_v b_1 \ldots b_v} F^{b_1 \ldots b_v}(\lambda), \tag{4.14}$$

where strengths for the gauge transformation parameters are defined as in (4.7).

From (4.12), (4.13), we see that the gauge fields $\phi_{-1}^{a_1 \ldots a_{v-1}}$, $\phi_0^{a_1 \ldots a_{v-2}}$ transform as Stueckelberg fields, i.e. these fields can be gauged away via Stueckelberg gauge fixing, $\phi_{-1}^{a_1 \ldots a_{v-1}} = 0$, $\phi_0^{a_1 \ldots a_{v-2}} = 0$. If we gauge away these fields and exclude the field $\phi_1^{a_1 \ldots a_{v-1}}$ via equations of motion, then our Lagrangians (4.5), (4.6) reduce to the respective Lagrangians of the standard approach, (2.29), (2.30). Thus, our approach is equivalent to the standard one. Note that one-derivative contributions to gauge transformations (4.11)–(4.14) take the form of the standard gauge transformations for antisymmetric tensor fields.

*4.2. Realization of conformal algebra symmetries*

To complete the ordinary-derivative description of the conformal self-dual field, we should provide the realization of the conformal algebra symmetries on a space of tensor fields (4.1). The realization of the Poincaré algebra symmetries on the tensor fields is well known. The realization of dilatation symmetry is given by (2.18), where conformal dimensions of fields (4.1) are given in (4.3). The general form of conformal boost transformations of arbitrary tensor fields is given in (2.23). According to (2.23), we should find realizations of the operators $K^a_{\Delta,M}$ and $R^a$ on space of fields (4.1). The realization of the former operator is obtained by adopting the general formula (2.24) for gauge fields (4.1). The real problem is getting a realization of the operator $R^a$. The realization of the operator $R^a$ on the space of gauge fields (4.1) that we found is given by

$$\delta_{R^a}\phi_1^{a_1\ldots a_{v-1}} = -t^{aa_1\ldots a_{v-1}} - (v-1)\eta^{a[a_1}\phi_0^{a_2\ldots a_{v-1}]}$$
$$- \partial^a\phi_{-1}^{a_1\ldots a_{v-1}} - \frac{\tau}{v!}\epsilon^{aa_1\ldots a_{v-1}b_1\ldots b_v}F^{b_1\ldots b_v}(\phi_{-1}), \tag{4.15}$$

$$\delta_{R^a}\phi_{-1}^{a_1\ldots a_{v-1}} = 0, \tag{4.16}$$

$$\delta_{R^a}\phi_0^{a_1\ldots a_{v-2}} = \phi_{-1}^{aa_1\ldots a_{v-2}}, \tag{4.17}$$

$$\delta_{R^a}t^{a_1\ldots a_v} = v\eta^{a[a_1}\phi_{-1}^{a_2\ldots a_v]} + \frac{\tau}{(v-1)!}\epsilon^{a_1\ldots a_v ab_1\ldots b_{v-1}}\phi_{-1}^{b_1\ldots b_{v-1}}. \tag{4.18}$$

Making use of these relations and general formula (2.23) gives the conformal boost transformations of gauge fields (4.1).

*4.3. On-shell degrees of freedom and the light-cone gauge Lagrangian*

We now discuss on-shell D.o.F. of the self-dual field. As before, for this purpose it is convenient to use fields transforming in irreps of the $so(d-2)$ algebra. Using the method in appendix C, one can prove that on-shell D.o.F. are described by the following antisymmetric tensor fields of the $so(d-2)$ algebra:

$$\phi_{-1}^{i_1\ldots i_{v-1}}, \qquad \phi_1^{i_1\ldots i_{v-1}}, \qquad \phi_0^{i_1\ldots i_{v-2}}, \tag{4.19}$$

where vector indices of the $so(d-2)$ algebra take values $i, j = 1, 2, \ldots d-2$. Fields in (4.19) are complex-valued when $d = 4k$ and real-valued when $d = 4k+2$. The fields $\phi_{-1}^{i_1\ldots i_{v-1}}$, $\phi_1^{i_1\ldots i_{v-1}}$ satisfy the $so(d-2)$ self-duality constraint:

$$\phi_{-1}^{i_1\ldots i_{v-1}} = \frac{\tau^{-1}}{(v-1)!}\epsilon^{i_1\ldots i_{v-1}j_1\ldots j_{v-1}}\phi_{-1}^{j_1\ldots j_{v-1}}, \tag{4.20}$$

$$\phi_1^{i_1\ldots i_{v-1}} = \frac{\tau^{-1}}{(v-1)!}\epsilon^{i_1\ldots i_{v-1}j_1\ldots j_{v-1}}\phi_1^{j_1\ldots j_{v-1}}, \tag{4.21}$$

where $\epsilon^{i_1\ldots i_{v-1}j_1\ldots j_{v-1}}$ is the Levi-Civita symbol normalized as $\epsilon^{12\ldots d-2} = 1$.

The total number of real-valued on-shell D.o.F. given in (4.19) is equal to

$$\mathbf{n} = \frac{h(5v-7)(2v-4)!}{(v-1)!(v-2)!}, \qquad h = \begin{cases} 2 & \text{for } d = 4k; \\ 1 & \text{for } d = 4k+2. \end{cases} \tag{4.22}$$

Namely, we note that $\mathbf{n}$ is a sum of $\mathbf{n}(\phi_{\pm 1}^{i_1\ldots i_{v-1}})$ and $\mathbf{n}(\phi_0^{i_1\ldots i_{v-2}})$ which are the respective numbers of the real-valued independent tensorial components of the fields $\phi_{\pm 1}^{i_1\ldots i_{v-1}}$ and $\phi_0^{i_1\ldots i_{v-2}}$:

$$\mathbf{n} = \mathbf{n}(\phi_1^{i_1\ldots i_{v-1}}) + \mathbf{n}(\phi_{-1}^{i_1\ldots i_{v-1}}) + \mathbf{n}(\phi_0^{i_1\ldots i_{v-2}}), \tag{4.23}$$

$$\mathbf{n}\left(\phi_{\pm 1}^{i_1 \ldots i_{\nu-1}}\right) = \frac{h(2\nu - 2)!}{2((\nu - 1)!)^2}, \qquad \mathbf{n}\left(\phi_0^{i_1 \ldots i_{\nu-2}}\right) = \frac{h(2\nu - 4)!}{((\nu - 2)!)^2}. \qquad (4.24)$$

Using the light-cone gauge frame, one can make sure that, for $d = 4k$, the gauge invariant Lagrangian (4.5) leads to the following light-cone gauge Lagrangian for fields (4.19):

$$\mathcal{L}_{\text{l.c.}} = \frac{1}{(\nu - 1)!} \bar{\phi}_1^{i_1 \ldots i_{\nu-1}} \square \phi_{-1}^{i_1 \ldots i_{\nu-1}} + \frac{1}{(\nu - 1)!} \bar{\phi}_{-1}^{i_1 \ldots i_{\nu-1}} \square \phi_1^{i_1 \ldots i_{\nu-1}}$$
$$+ \frac{1}{(\nu - 2)!} \bar{\phi}_0^{i_1 \ldots i_{\nu-2}} \square \phi_0^{i_1 \ldots i_{\nu-2}} - \frac{1}{(\nu - 1)!} \bar{\phi}_1^{i_1 \ldots i_{\nu-1}} \phi_1^{i_1 \ldots i_{\nu-1}}, \qquad (4.25)$$

while, for $d = 4k + 2$, the gauge invariant Lagrangian (4.6) leads to the following light-cone gauge Lagrangian for fields (4.19):

$$\mathcal{L}_{\text{l.c.}} = \frac{1}{(\nu - 1)!} \phi_1^{i_1 \ldots i_{\nu-1}} \square \phi_{-1}^{i_1 \ldots i_{\nu-1}} + \frac{1}{2(\nu - 2)!} \phi_0^{i_1 \ldots i_{\nu-2}} \square \phi_0^{i_1 \ldots i_{\nu-2}}$$
$$- \frac{1}{2(\nu - 1)!} \phi_1^{i_1 \ldots i_{\nu-1}} \phi_1^{i_1 \ldots i_{\nu-1}}. \qquad (4.26)$$

## 5. Oscillator form of the Lagrangian

In the preceding sections, we have presented our results for the self-dual fields by using the representation of the field content in terms of the tensor fields. However, the use of such representation is not convenient in many applications. In this section, we represent our results by using the representation of the field content in terms of generating functions constructed out of the tensor fields and the appropriate oscillators[7]. This is to say that in order to obtain the Lagrangian description in an easy-to-use form, we introduce creation operators $\alpha^a$, $\zeta$, $\upsilon^\oplus$, $\upsilon^\ominus$ and the respective annihilation operators $\bar{\alpha}^a$, $\bar{\zeta}$, $\bar{\upsilon}^\ominus$, $\bar{\upsilon}^\oplus$ and collect tensor fields (4.1) in the ket-vector $|\Phi\rangle$ defined by

$$|\Phi\rangle = \begin{pmatrix} |\phi\rangle \\ |t\rangle \end{pmatrix}, \qquad (5.1)$$

$$|\phi\rangle = \upsilon^\oplus |\phi_1\rangle + \upsilon^\ominus |\phi_{-1}\rangle + \zeta |\phi_0\rangle, \qquad (5.2)$$

$$|\phi_{k'}\rangle \equiv \frac{1}{(\nu - 1)!} \alpha^{a_1} \ldots \alpha^{a_{\nu-1}} \phi_{k'}^{a_1 \ldots a_{\nu-1}} |0\rangle, \qquad k' = -1, 1, \qquad (5.3)$$

$$|\phi_0\rangle \equiv \frac{1}{(\nu - 2)!} \alpha^{a_1} \ldots \alpha^{a_{\nu-2}} \phi_0^{a_1 \ldots a_{\nu-2}} |0\rangle, \qquad (5.4)$$

$$|t\rangle \equiv \frac{1}{\nu!} \alpha^{a_1} \ldots \alpha^{a_\nu} t^{a_1 \ldots a_\nu} |0\rangle. \qquad (5.5)$$

In the literature, ket-vectors (5.1)–(5.5) are sometimes referred to as generating functions. The ket-vectors $|\phi\rangle$, $|t\rangle$ satisfy the obvious algebraic constraints

$$(N_\alpha + N_\zeta)|\phi\rangle = (\nu - 1)|\phi\rangle, \qquad (5.6)$$

$$(N_\zeta + N_\upsilon)|\phi\rangle = |\phi\rangle, \qquad (5.7)$$

$$N_\alpha |t\rangle = \nu |t\rangle, \qquad (5.8)$$

---

[7] Note that in this paper we use oscillators just to handle the many indices appearing for tensor fields. In a proper way, the oscillators arise in the framework of the world-line approach to higher spin fields (see e.g. [17–19]).

$$N_\zeta |t\rangle = 0, \qquad N_\upsilon |t\rangle = 0, \tag{5.9}$$

where we use the notation given in (2.6)–(2.10). We note that these algebraic constraints tell us about the number of oscillators $\alpha^a$, $\zeta$, $\upsilon^\oplus$, $\upsilon^\ominus$ appearing in the ket-vectors $|\phi\rangle$ and $|t\rangle$. In terms of the ket-vector $|t\rangle$, the self-duality constraint (4.2) takes the form

$$|t\rangle = \epsilon |t\rangle, \tag{5.10}$$

where we use the notation for the $\epsilon$-symbol

$$\epsilon \equiv \frac{\tau}{(\nu!)^2} \epsilon^{a_1 \ldots a_\nu b_1 \ldots b_\nu} \alpha^{a_1} \ldots \alpha^{a_\nu} \bar{\alpha}^{b_\nu} \ldots \bar{\alpha}^{b_1}. \tag{5.11}$$

Useful relations for the $\epsilon$-symbol (5.11) and various related $\epsilon$-symbols may be found in appendix D.

In terms of the ket-vector $|\Phi\rangle$, Lagrangians (4.5), (4.6) can be re-expressed as

$$\mathcal{L} = \frac{h}{2} \langle \Phi | E | \Phi \rangle, \tag{5.12}$$

where the normalization factor $h$ is given in (4.22), while the operator $E$ is defined by the relations

$$E = \begin{pmatrix} E_{\phi\phi} & E_{\phi t} \\ E_{t\phi} & 0 \end{pmatrix}, \tag{5.13}$$

$$E_{\phi\phi} = E_{\phi\phi(2)} + E_{\phi\phi(1)} + E_{\phi\phi(0)}, \tag{5.14}$$

$$E_{\phi t} = \upsilon^\ominus \bar{\alpha} \partial, \qquad E_{t\phi} = -\bar{\upsilon}^\ominus \alpha \partial, \tag{5.15}$$

$$E_{\phi\phi(2)} = \Box - \alpha \partial \bar{\alpha} \partial, \tag{5.16}$$

$$E_{\phi\phi(1)} = e_1 \bar{\alpha} \partial + \alpha \partial \bar{e}_1, \tag{5.17}$$

$$E_{\phi\phi(0)} = m_1, \tag{5.18}$$

$$e_1 = \zeta \bar{\upsilon}^\ominus, \qquad \bar{e}_1 = -\upsilon^\ominus \bar{\zeta}, \qquad m_1 = \upsilon^\ominus \bar{\upsilon}^\ominus (N_\zeta - 1). \tag{5.19}$$

Alternatively, the Lagrangian (5.12) can be represented in terms of the ket-vector of gauge fields (5.2)–(5.5). This is to say that the Lagrangian takes the form (up to a total derivative)

$$-\mathcal{L} = \langle F(\phi_{-1}) | F(\phi_1) \rangle + \langle F(\phi_1) | F(\phi_{-1}) \rangle + \langle F(\phi_1) | t \rangle + \langle t | F(\phi_1) \rangle$$
$$+ (\langle \phi_1 | + \langle F(\phi_0) |)(|\phi_1\rangle + |F(\phi_0)\rangle), \tag{5.20}$$

when $d = 4k$, and

$$-\mathcal{L} = \langle F(\phi_{-1}) | F(\phi_1) \rangle + \langle F(\phi_1) | t \rangle + \tfrac{1}{2}(\langle \phi_1 | + \langle F(\phi_0) |)(|\phi_1\rangle + |F(\phi_0)\rangle), \tag{5.21}$$

when $d = 4k + 2$. In (5.20), (5.21) and below, the ket-vector of the field strength $|F(\phi)\rangle$ is defined as

$$|F(\phi)\rangle = \alpha \partial |\phi\rangle. \tag{5.22}$$

### 5.1. Gauge transformations

Gauge transformations can also be cast into the generating form. To this end we use, as before, the oscillators $\alpha^a$, $\zeta$, $\upsilon^\oplus$, $\upsilon^\ominus$ and collect gauge transformation parameters (4.9) into the ket-vector $|\Xi\rangle$ defined by

$$|\Xi\rangle = \begin{pmatrix} |\xi\rangle \\ |\lambda\rangle \end{pmatrix}, \tag{5.23}$$

where the ket-vectors $|\xi\rangle$ and $|\lambda\rangle$ are defined as

$$|\xi\rangle = \upsilon^{\oplus}|\xi_0\rangle + \upsilon^{\ominus}|\xi_{-2}\rangle - \zeta|\xi_{-1}\rangle, \tag{5.24}$$

$$|\xi_{k'-1}\rangle \equiv \frac{1}{(\nu - 2)!}\alpha^{a_1}\ldots\alpha^{a_{\nu-2}}\xi_{k'-1}^{a_1\ldots a_{\nu-2}}|0\rangle, \qquad k' = -1, 1, \tag{5.25}$$

$$|\xi_{-1}\rangle \equiv \frac{1}{(\nu - 3)!}\alpha^{a_1}\ldots\alpha^{a_{\nu-3}}\xi_{-1}^{a_1\ldots a_{\nu-3}}|0\rangle, \tag{5.26}$$

$$|\lambda\rangle \equiv \frac{1}{(\nu - 1)!}\alpha^{a_1}\ldots\alpha^{a_{\nu-1}}\lambda^{a_1\ldots a_{\nu-1}}|0\rangle. \tag{5.27}$$

The ket-vectors $|\xi\rangle$, $|\lambda\rangle$ satisfy the obvious algebraic constraints

$$(N_\alpha + N_\zeta)|\xi\rangle = (\nu - 2)|\xi\rangle, \tag{5.28}$$

$$(N_\zeta + N_\upsilon)|\xi\rangle = |\xi\rangle, \tag{5.29}$$

$$N_\alpha|\lambda\rangle = (\nu - 1)|\lambda\rangle, \tag{5.30}$$

$$N_\zeta|\lambda\rangle = 0, \qquad N_\upsilon|\lambda\rangle = 0. \tag{5.31}$$

As before, these constraints tell us about the number of oscillators $\alpha^a$, $\zeta$, $\upsilon^{\oplus}$, $\upsilon^{\ominus}$ appearing in the ket-vectors $|\xi\rangle$ and $|\lambda\rangle$.

Now, gauge transformations (4.11)–(4.14) can entirely be represented in terms of the ket-vectors $|\Phi\rangle$ and $|\Xi\rangle$:

$$\delta|\Phi\rangle = G|\Xi\rangle, \tag{5.32}$$

where the operator $G$ is given by

$$G = \begin{pmatrix} \alpha\partial - \zeta\bar{\upsilon}^{\ominus} & -\upsilon^{\ominus} \\ 0 & (1 + \epsilon)\alpha\partial \end{pmatrix}, \tag{5.33}$$

and the $\epsilon$-symbol is defined in (5.11).

Alternatively, the gauge transformation (5.32) can be represented in terms of the ket-vector of gauge fields (5.2)–(5.5). This is to say that the gauge transformation (5.32) amounts to the following gauge transformations:

$$\delta|\phi_1\rangle = |F(\xi_0)\rangle, \tag{5.34}$$

$$\delta|\phi_{-1}\rangle = |F(\xi_{-2})\rangle - |\lambda\rangle, \tag{5.35}$$

$$\delta|\phi_0\rangle = |F(\xi_{-1})\rangle - |\xi_0\rangle, \tag{5.36}$$

$$\delta|t\rangle = (1 + \epsilon)|F(\lambda)\rangle, \tag{5.37}$$

where the field strengths for the ket-vectors of the gauge transformation parameters are defined as in (5.22).

## 5.2. Oscillator realization of conformal algebra symmetries on gauge fields

To complete the oscillator description of the conformal self-dual field, we provide a realization of the conformal algebra symmetries on space of the ket-vector $|\Phi\rangle$. The realization of the Poincaré algebra symmetries and dilatation symmetry is given by (2.16)–(2.18), where the operators $M^{ab}$ and $\Delta$ take the form

$$M^{ab} = \alpha^a\bar{\alpha}^b - \alpha^b\bar{\alpha}^a, \tag{5.38}$$

$$\Delta = \frac{d-2}{2} + \Delta', \qquad \Delta' \equiv N_{\upsilon^\oplus} - N_{\upsilon^\ominus}. \tag{5.39}$$

Conformal boost transformations of $|\Phi\rangle$ are given in (2.19). According to (2.19), we should find the operators $K^a_{\Delta,M}$ and $R^a$. The former operator is given in (2.20). The realization of the operator $R^a$ on the space of $|\Phi\rangle$ can be read from (4.15)–(4.18). Namely, in terms of the ket-vector $|\Phi\rangle$, transformations of the gauge fields given in (4.15)–(4.18) can be represented as

$$\delta_{R^a}|\Phi\rangle = R^a|\Phi\rangle, \tag{5.40}$$

where the realization of the operator $R^a$ on $|\Phi\rangle$ takes the form

$$R^a = \begin{pmatrix} R^a_{\phi\phi} & R^a_{\phi t} \\ R^a_{t\phi} & 0 \end{pmatrix}, \tag{5.41}$$

$$R^a_{\phi\phi} = r_{0,1}\bar{\alpha}^a + \alpha^a \bar{r}_{0,1} + r_{1,1}\left(\eta^{ab} + \epsilon^{ab}_0\right)\partial^b, \tag{5.42}$$

$$R^a_{\phi t} = r_{0,4}\bar{\alpha}^a, \tag{5.43}$$

$$R^a_{t\phi} = r_{0,5}(1+\epsilon)\alpha^a, \tag{5.44}$$

$$r_{0,1} = \zeta\bar{\upsilon}^\oplus, \qquad \bar{r}_{0,1} = -\upsilon^\oplus\bar{\zeta}, \tag{5.45}$$

$$r_{1,1} = -\upsilon^\oplus\bar{\upsilon}^\oplus, \tag{5.46}$$

$$r_{0,4} = -\upsilon^\oplus, \qquad r_{0,5} = \bar{\upsilon}^\oplus, \tag{5.47}$$

and the $\epsilon^{ab}_0$-symbol is defined in (D.4).

Alternatively, conformal boost transformations (5.40) can be represented in terms of ket-vectors (5.2)–(5.5). To this end, we note that the realization of the operator $K^a_{\Delta,M}$ on the space of ket-vectors (5.2)–(5.5) is given by (2.20), where the operators $M^{ab}$ and $\Delta$ take the same form as in (5.38), (5.39). Note that (5.39) implies the following conformal dimensions of the respective ket-vectors:

$$\Delta(|\phi_{k'}\rangle) = \frac{d-2}{2} + k', \qquad k' = 0, \pm 1, \tag{5.48}$$

$$\Delta(|t\rangle) = \frac{d-2}{2}. \tag{5.49}$$

We now make sure that the realization of the operator $R^a$ given in (5.40) can be represented in terms of ket-vectors (5.2)–(5.5) as

$$\delta_{R^a}|\phi_1\rangle = -\bar{\alpha}^a|t\rangle - \alpha^a|\phi_0\rangle - \left(\eta^{ab} + \epsilon^{ab}_0\right)\partial^b|\phi_{-1}\rangle, \tag{5.50}$$

$$\delta_{R^a}|\phi_{-1}\rangle = 0, \tag{5.51}$$

$$\delta_{R^a}|\phi_0\rangle = \bar{\alpha}^a|\phi_{-1}\rangle, \tag{5.52}$$

$$\delta_{R^a}|t\rangle = (1+\epsilon)\alpha^a|\phi_{-1}\rangle. \tag{5.53}$$

Making use of these relations gives the conformal boost transformations of ket-vectors (5.2)–(5.5).

We now recall that the realization of the conformal symmetries on a space of the ket-vectors of gauge fields (5.50)–(5.53) is defined up to gauge transformations (5.34)–(5.37). As we have demonstrated in section 4, the conformal symmetries can also be realized on a space of the field strengths. We now discuss the oscillator form of the field strengths for gauge fields (5.2)–(5.5) and the corresponding realization of conformal symmetries on a space of the field strengths.

### 5.3. Oscillator realization of conformal algebra symmetries on field strengths

We introduce the following ket-vectors of field strengths constructed out of ket-vectors of gauge fields (5.2)–(5.5):

$$|F^\ominus\rangle = |t\rangle + (1 + \epsilon)|F(\phi_{-1})\rangle, \tag{5.54}$$

$$|F\rangle = \alpha\partial|F^\ominus\rangle, \tag{5.55}$$

$$|F^{\oplus\ominus}\rangle = |\phi_1\rangle + |F(\phi_0)\rangle, \tag{5.56}$$

$$|F^\oplus\rangle = |F(\phi_1)\rangle, \tag{5.57}$$

where $|F(\phi_{k'})\rangle$, $k' = 0, \pm 1$ are defined as in (5.22). One can make sure that field strengths (5.54)–(5.57) are invariant under gauge transformations (5.34)–(5.37).

Conformal dimensions of the field strengths can be read from (5.48), (5.49) and (5.54)–(5.57):

$$\Delta(|F^\ominus\rangle) = \frac{d-2}{2}, \quad \Delta(|F\rangle) = \frac{d}{2}, \quad \Delta(|F^{\oplus\ominus}\rangle) = \frac{d}{2}, \quad \Delta(|F^\oplus\rangle) = \frac{d+2}{2}. \tag{5.58}$$

The realization of the Poincaré algebra symmetries and dilatation symmetry on the space of the ket-vectors of field strengths is given by (2.16)–(2.18), where the operator $M^{ab}$ takes the form as in (5.38), while conformal dimensions are given in (5.58).

Making use of transformations of the ket-vectors of gauge fields given in (5.50)–(5.53), we find the corresponding conformal boost transformations of the ket-vectors of field strengths,

$$\delta_{K^a}|F^\ominus\rangle = K^a_{\Delta,M}|F^\ominus\rangle, \tag{5.59}$$

$$\delta_{K^a}|F\rangle = K^a_{\Delta,M}|F\rangle - \alpha^a|F^\ominus\rangle, \tag{5.60}$$

$$\delta_{K^a}|F^{\oplus\ominus}\rangle = K^a_{\Delta,M}|F^{\oplus\ominus}\rangle - \bar{\alpha}^a|F^\ominus\rangle, \tag{5.61}$$

$$\delta_{K^a}|F^\oplus\rangle = K^a_{\Delta,M}|F^\oplus\rangle + \alpha^a|F^{\oplus\ominus}\rangle + \bar{\alpha}^a|F\rangle - \partial^a|F^\ominus\rangle, \tag{5.62}$$

where the operator $K^a_{\Delta,M}$ is given in (2.20), while the conformal dimensions are defined in (5.58). Comparing these formulas with the general relation (2.19), we find the realization of the operator $R^a$ on a space of the ket-vectors of the field strengths:

$$\delta_{R^a}|F^\ominus\rangle = 0, \tag{5.63}$$

$$\delta_{R^a}|F\rangle = -\alpha^a|F^\ominus\rangle, \tag{5.64}$$

$$\delta_{R^a}|F^{\oplus\ominus}\rangle = -\bar{\alpha}^a|F^\ominus\rangle, \tag{5.65}$$

$$\delta_{R^a}|F^\oplus\rangle = \alpha^a|F^{\oplus\ominus}\rangle + \bar{\alpha}^a|F\rangle - \partial^a|F^\ominus\rangle. \tag{5.66}$$

### 5.4. Oscillator form of the light-cone gauge Lagrangian

To discuss the oscillator form of the light-cone gauge Lagrangian, we collect fields (4.19) into the following ket-vectors:

$$|\phi_{\text{l.c.}}\rangle = \upsilon^\oplus|\phi_1\rangle_{\text{l.c}} + \upsilon^\ominus|\phi_{-1}\rangle_{\text{l.c}} + \zeta|\phi_0\rangle_{\text{l.c}}, \tag{5.67}$$

$$|\phi_{k'}\rangle_{\text{l.c}} \equiv \frac{1}{(\nu-1)!}\alpha^{i_1}\ldots\alpha^{i_{\nu-1}}\phi_{k'}^{i_1\ldots i_{\nu-1}}|0\rangle, \qquad k' = -1, 1, \tag{5.68}$$

$$|\phi_0\rangle_{\text{l.c}} \equiv \frac{1}{(\nu-2)!}\alpha^{i_1}\ldots\alpha^{i_{\nu-2}}\phi_0^{i_1\ldots i_{\nu-2}}|0\rangle. \tag{5.69}$$

In terms of the ket-vector $|\phi_{\text{l.c.}}\rangle$ (5.67), the light-cone gauge Lagrangians (4.25), (4.26) can concisely be represented as

$$\mathcal{L}_{\text{l.c.}} = \frac{h}{2}\langle\phi_{\text{l.c.}}|(\Box - \mathcal{M}^2)|\phi_{\text{l.c.}}\rangle, \tag{5.70}$$

$$\mathcal{M}^2 \equiv \upsilon^\ominus\bar{\upsilon}^\ominus, \tag{5.71}$$

where the normalization factor $h$ is given in (4.22).

## 6. Conclusions

In this paper, we applied the ordinary-derivative approach, developed in [25], to the study of conformal self-dual fields in the flat space of an even dimension. The results presented here should have a number of interesting applications and generalizations, some of which are as follows.

(i) The results in this paper and the ones in [25] provide the complete ordinary-derivative description of all fields that appear in the graviton supermultiplets of conformal supergravity theories. It would be interesting to apply these results to the study of supersymmetric conformal field theories [20–24] in the framework of ordinary-derivative approach. The first step in this direction would be the understanding of how the supersymmetries are realized in the framework of our approach.

(ii) Our approach to conformal theories (see [25, 26]) is based on the new realization of conformal gauge symmetries via Stueckelberg fields. In our approach, the use of the Stueckelberg fields is very similar to the one in the gauge invariant formulation of massive fields. Stueckelberg fields provide interesting possibilities for the study of interacting massive gauge fields (see e.g. [27, 28]). So we think that application of our approach to the interacting conformal self-dual fields may lead to new interesting development.

(iii) The BRST approach is one of the powerful approaches to the analysis of various aspects of relativistic dynamics (see e.g. [29, 30]). This approach is conveniently adapted for the ordinary-derivative formulation. Recent application of the BRST approach to the study of totally antisymmetric fields may be found in [31]. We think that extension of this approach to the case of conformal self-dual fields should be relatively straightforward.

(iv) Self-dual fields studied in this paper are the particular case of mixed-symmetry fields. In the previous years, there were interesting developments in studying the mixed-symmetry fields [32–35] that are invariant with respect to Poincaré algebra symmetries. It would be interesting to apply methods developed in [32–35] to studying the conformal self-dual mixed-symmetry fields[8]. There are various other interesting approaches in the literature which could be used to discuss the ordinary-derivative formulation of conformal self-dual fields. This is to say that various recently developed interesting formulations in terms of unconstrained fields in flat space may be found in [38, 39].

### Acknowledgments

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[8] The unfolded form of equations of motion for conformal mixed-symmetry fields is studied in [36]. Higher-derivative Lagrangian formulation of the mixed-symmetry conformal fields was recently developed in [37].

## Appendix A. Derivation of the ordinary-derivative gauge invariant Lagrangian

Because the methods for finding the ordinary-derivative Lagrangian for arbitrary $d \geqslant 4$ are quite similar we present details of the derivation for the case of $d = 4$. To derive the ordinary-derivative gauge invariant Lagrangian (3.5), we use the Lagrangian of the standard formulation given in (2.33). First, in place of the field $T^{ab}$, we introduce the fields $t^{ab}$ and $\phi^a_{-1}$ by the relation

$$T^{ab} = t^{ab} + F^{ab}(\phi_{-1}) + \frac{\tau}{2}\epsilon^{abce}F^{ce}(\phi_{-1}),\tag{A.1}$$

where $t^{ab}$ satisfies the self-duality constraint (3.2). Plugging (A.1) in (2.33) we obtain

$$\mathcal{L}_{\text{st}} = -\tfrac{1}{2}F^{ab}(\bar{\phi}_{-1})\,\square\,F^{ab}(\phi_{-1})$$

$$-\tfrac{1}{2}\bar{t}^{ab}\,\square\,F^{ab}(\phi_{-1}) - \tfrac{1}{2}F^{ab}(\bar{\phi}_{-1})\,\square\,t^{ab} + \partial^a\bar{t}^{ac}\partial^b t^{bc},\tag{A.2}$$

where the field strength $F^{ab}(\phi_{-1})$ is defined as in (3.6). We note that the representation for $T^{ab}$ given in (A.1) implies that $T^{ab}$ is invariant under the gauge transformations

$$\delta\phi^a_{-1} = \partial^a\xi_{-2} - \lambda^a,\tag{A.3}$$

$$\delta t^{ab} = \partial^a\lambda^b - \partial^b\lambda^a + \tau\epsilon^{abce}\partial^c\lambda^e.\tag{A.4}$$

This implies that Lagrangian (A.2) is also invariant under gauge transformations (A.3), (A.4).

Second, we introduce new fields $\phi^a_1$ and $\phi_0$ by using the following Lagrangian in place of (A.2):

$$\mathcal{L} = \mathcal{L}_{\text{st}} - \bar{X}^a X^a,\tag{A.5}$$

where we use the notation

$$X^a \equiv \phi^a_1 + \partial^a\phi_0 - \partial^b F^{ba}(\phi_{-1}) - \partial^b t^{ba},\tag{A.6}$$

and $\bar{X}^a$ is complex conjugate of $X^a$. It is clear that, on-shell, Lagrangians $\mathcal{L}_{\text{st}}$ and $\mathcal{L}$ describe the same field D.o.F. Using the formula (up to total derivative)

$$\partial^a t^{ac}\partial^b F^{bc} = -\tfrac{1}{2}t^{ab}\,\square\,F^{ab},\tag{A.7}$$

it is easy to see that Lagrangian (A.5) gives the ordinary-derivative Lagrangian (3.5).

We now consider gauge symmetries of Lagrangian (A.5). Because the contribution to $X^a$ given by (see (A.6))

$$-\,\partial^b F^{ba}(\phi_{-1}) - \partial^b t^{ba}\tag{A.8}$$

is invariant under gauge transformations (A.3), (A.4), the $X^a$ is also invariant under these gauge transformations, i.e. Lagrangian (A.5) is invariant under gauge transformations (A.3), (A.4). Besides this, we note that $X^a$ is invariant under the additional gauge transformations

$$\delta\phi^a_1 = \partial^a\xi_0, \qquad \delta\phi_0 = -\xi_0.\tag{A.9}$$

Altogether, gauge transformations (A.3), (A.4), (A.9) amount to the ones given in (3.10)–(3.13).

## Appendix B. Derivation of the operator $R^a$

In this appendix, we outline the derivation of the operator $R^a$ (5.41). The use of the oscillator formulation turns out to be convenient for this purpose. The operator $R^a$ is then determined by requiring the action of the self-dual field (4.4) to be invariant under the conformal boost transformations. In order to analyze restrictions imposed on the operator $R^a$ by the conformal

boost symmetries, we need to know the explicit form of the restrictions imposed on the operator $E$ (5.13) by the Lorentz and dilatation symmetries. Requiring action (4.4) with Lagrangian (5.12) to be invariant under the Lorentz and dilatation symmetries $\delta_{J^{ab}} S = 0$, $\delta_D S = 0$, amounts to the following respective equations for the operator $E$:

$$[E, J^{ab}] = 0, \qquad [E, D] = 2E, \tag{B.1}$$

where the operators $J^{ab}$ and $D$ are given in (2.17) and (2.18), respectively.

The variation of Lagrangian (5.12) under the conformal boost transformation can be presented as (up to total derivative)

$$\delta_{K^a}\mathcal{L} = \frac{h}{2}\langle\Phi|\delta_{K^a}E|\Phi\rangle, \tag{B.2}$$

$$\delta_{K^a}E \equiv K^{a\dagger}E + EK^a. \tag{B.3}$$

Using (B.1) and $K^a$ given in (2.19), we make sure that $\delta_{K^a}E$ (B.3) can be represented as

$$\delta_{K^a}E = R^{a\dagger}E + ER^a + E^a, \tag{B.4}$$

$$E^a \equiv (\Delta + 1)[E, x^a] + [E, x^b]M^{ab} - \tfrac{1}{2}[[E, x^b], x^b]\partial^a, \tag{B.5}$$

where $M^{ab}$ and $\Delta$ are given in (5.38), (5.39). From (B.2), (B.4), we see that the requirement of invariance of the action under the conformal boost transformations amounts to the equations

$$R^{a\dagger}E + ER^a + E^a \approx 0. \tag{B.6}$$

In (B.6) and below, to simplify our formulas, we adopt the following convention. Let $A$ be some operator. We use the relation $A \approx 0$ in place of $\langle\Phi|A|\Phi\rangle = 0$.

Using the operator $E$ (5.13)–(5.19) and formula (B.5), we find immediately the operator $E^a$:

$$E^a = \begin{pmatrix} E^a_{\phi\phi(1)} + E^a_{\phi\phi(0)} & -\upsilon^\ominus\bar{\alpha}^a \\ -\alpha^a\bar{\upsilon}^\ominus & 0 \end{pmatrix}, \tag{B.7}$$

$$E^a_{\phi\phi(1)} = 2\Delta'\partial^a - (\Delta' + N_\zeta)\alpha^a\bar{\alpha}\partial + (-\Delta' + N_\zeta)\alpha\partial\bar{\alpha}^a, \tag{B.8}$$

$$E^a_{\phi\phi(0)} = (\Delta' - N_\zeta)e_1\bar{\alpha}^a + \alpha^a\bar{e}_1(\Delta' + N_\zeta), \tag{B.9}$$

where $e_1$, $\bar{e}_1$ and $\Delta'$ are given in (5.19) and (5.39), respectively. Also we note that the commutation relation $[D, K^a] = K^a$ gives the following equation for the operator $R^a$:

$$[D, R^a] = R^a. \tag{B.10}$$

Equations (B.6), (B.10) constitute a complete system of equations which allows us to determine the operator $R^a$ uniquely. We now discuss the procedure of solving these equations.

The operator $E$ (5.13) is a second-order polynomial in the derivative. From (B.7), we see that the operator $E^a$ is a first-order polynomial in the derivative. The operator $R^a$ also turns out to be the first-order polynomial in the derivative. Therefore, it is convenient to represent the operators $E$, $E^a$ and $R^a$ as power series in the derivative:

$$E = E_{(2)} + E_{(1)} + E_{(0)}, \tag{B.11}$$

$$E^a = E^a_{(1)} + E^a_{(0)}, \qquad R^a = R^a_{(1)} + R^a_{(0)}, \tag{B.12}$$

where the operators $E_{(n)}$, $E^a_{(n)}$ and $R^a_{(n)}$ are degree-$n$ homogeneous polynomials in the derivative. Explicit expressions for the operators $E_{(n)}$ and $E^a_{(n)}$ can easily be read from the respective

expressions in (5.13)–(5.19) and (B.7)–(B.9). Using (B.11) and (B.12), it is easy to see that equation (B.6) amounts to the following equations:

$$E_{(2)} R^a_{(1)} + \text{h.c.} \approx 0, \tag{B.13}$$

$$E_{(2)} R^a_{(0)} + E_{(1)} R^a_{(1)} + \text{h.c.} \approx 0, \tag{B.14}$$

$$\left( E_{(1)} R^a_{(0)} + E_{(0)} R^a_{(1)} + \text{h.c.} \right) + E^a_{(1)} \approx 0, \tag{B.15}$$

$$\left( E_{(0)} R^a_{(0)} + \text{h.c.} \right) + E^a_{(0)} \approx 0. \tag{B.16}$$

We now present our procedure for solving equations (B.10) and (B.13)–(B.16).

(i) We note that the most general operators $R^a_{(n)}$ (B.12) acting on 2-vector $|\Phi\rangle$ (5.1) can be presented as $2 \times 2$ matrices given by

$$R^a_{(n)} = \begin{pmatrix} R^a_{\phi\phi(n)} & R^a_{\phi t(n)} \\ R^a_{t\phi(n)} & R^a_{tt(n)} \end{pmatrix}, \qquad n = 0, 1. \tag{B.17}$$

Requiring the operator $R^a$ to satisfy equation (B.10) and constraints (5.28)–(5.31), we find the following expressions[9]:

$$R^a_{\phi\phi(0)} = r_{0,1}\bar{\alpha}^a + \alpha^a \bar{r}_{0,1}, \tag{B.18}$$

$$R^a_{\phi\phi(1)} = r_{1,1}\partial^a + r_{1,5}\alpha^a\bar{\alpha}\partial + r_{\epsilon 1}\epsilon^{ab}_0\partial^b, \tag{B.19}$$

$$R^a_{\phi t(0)} = r_{0,4}\bar{\alpha}^a, \tag{B.20}$$

$$R^a_{\phi t(1)} = 0, \tag{B.21}$$

$$R^a_{t\phi(0)} = r_{0,5}(1 + \epsilon)\alpha^a, \tag{B.22}$$

$$R^a_{t\phi(1)} = 0, \tag{B.23}$$

$$R^a_{tt(n)} = 0, \qquad n = 0, 1, \tag{B.24}$$

where the operators $r_{0,1}, r_{0,2}, r_{0,4}, r_{0,5}, r_{1,1}, r_{1,5}, r_{\epsilon 1}$ independent of the oscillators $\alpha^a$ are given by

$$r_{0,1} = \zeta\tilde{r}_{0,1}\bar{\upsilon}^\oplus, \qquad \bar{r}_{0,1} = \upsilon^\oplus\tilde{\bar{r}}_{0,1}\bar{\zeta}, \tag{B.25}$$

$$r_{1,1} = \upsilon^\oplus\tilde{r}_{1,1}\bar{\upsilon}^\oplus, \qquad r_{\epsilon,1} = \upsilon^\oplus\tilde{r}_{\epsilon,1}\bar{\upsilon}^\oplus, \qquad r_{1,5} = \upsilon^\oplus\tilde{r}_{1,5}\bar{\upsilon}^\oplus, \tag{B.26}$$

$$r_{0,4} = \upsilon^\oplus\tilde{r}_{0,4}, \qquad r_{0,5} = \tilde{r}_{0,5}\bar{\upsilon}^\oplus, \tag{B.27}$$

and the quantities $\tilde{r}_{0,1}, \tilde{\bar{r}}_{0,1}, \tilde{r}_{0,4}, \tilde{r}_{0,5}, \tilde{r}_{1,1}, \tilde{r}_{1,5}, \tilde{r}_{\epsilon 1}$ independent of the oscillators $\alpha^a$, $\zeta$, $\upsilon^\oplus$, $\upsilon^\ominus$ remain as undermined constants. Below, we determine these quantities by using equations (B.13)–(B.16).

Before analyzing equations (B.13)–(B.16), we explain our terminology. Introducing the notation $\mathcal{X}$ for the left-hand side of equations (B.13)–(B.16), we note that $\mathcal{X}$ is a $2 \times 2$ matrix acting on 2-vector $|\Phi\rangle$ (5.1). Using the notation

$$\mathcal{X} = \begin{pmatrix} \mathcal{X}_{\phi\phi} & \mathcal{X}_{\phi t} \\ \mathcal{X}_{t\phi} & \mathcal{X}_{tt} \end{pmatrix}, \tag{B.28}$$

we note that the $2 \times 2$ matrix equation $\mathcal{X} \approx 0$ amounts to the four equations, $\langle\phi|\mathcal{X}_{\phi\phi}|\phi\rangle = 0$, $\langle\phi|\mathcal{X}_{\phi t}|t\rangle = 0$, $\langle t|\mathcal{X}_{t\phi}|\phi\rangle = 0$, $\langle t|\mathcal{X}_{tt}|t\rangle = 0$. We refer to these four equations as the respective $\phi\phi$-, $\phi t$-, $t\phi$- and $tt$-parts of the equation $\mathcal{X} \approx 0$. We now turn to the analysis of equations (B.13)–(B.16).

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[9]  As a realization of the operator $R^a$ on the gauge field $|\Phi\rangle$ is defined up to gauge transformation (5.32), we ignore contributions to $R^a$ that can be removed by the gauge transformation (5.32).

(ii) Using the $\phi\phi$ part of equation (B.13) and the relation

$$E_{\phi\phi(2)}R^a_{\phi\phi(1)} = r_{1,1}E_{\phi\phi(2)}\partial^a + r_{1,5}\Box\alpha^a\bar{\alpha}\partial - r_{1,5}\alpha\partial\bar{\alpha}\partial\partial^a + r_{\epsilon,1}\Box\epsilon_0^{ab}\partial^b, \qquad (B.29)$$

we find

$$\widetilde{r}^{\dagger}_{1,1} = \widetilde{r}_{1,1}, \qquad \widetilde{r}^{\dagger}_{\epsilon,1} = \widetilde{r}_{\epsilon,1}, \qquad \widetilde{r}_{1,5} = 0. \qquad (B.30)$$

(iii) Making use of the $\phi\phi$ part of equation (B.14) and the relations

$$E_{\phi\phi(2)}R^a_{\phi\phi(0)} = \alpha^a E_{\phi\phi(2)}\bar{r}_{0,1} + r_{0,1}E_{\phi\phi(2)}\bar{\alpha}^a - \alpha\partial\partial^a\bar{r}_{0,1}, \qquad (B.31)$$

$$E_{\phi\phi(1)}R^a_{\phi\phi(1)} = e_1 r_{1,1}\bar{\alpha}\partial\partial^a + \alpha\partial\partial^a\bar{e}_1 r_{1,1}, \qquad (B.32)$$

we obtain

$$\bar{r}_{0,1} = [\bar{e}_1, r_{1,1}], \qquad \widetilde{r}^{\dagger}_{0,1} = -\widetilde{\bar{r}}_{0,1}. \qquad (B.33)$$

(iv) Using the $\phi\phi$ part of equation (B.15) and the relations

$$E_{\phi\phi(1)}R^a_{\phi\phi(0)} = e_1\bar{r}_{0,1}\partial^a - e_1\bar{r}_{0,1}\alpha^a\bar{\alpha}\partial + \bar{e}_1 r_{0,1}\alpha\partial\bar{\alpha}^a, \qquad (B.34)$$

$$E_{\phi\phi(0)}R^a_{\phi\phi(1)} = m_1 r_{1,1}\partial^a + m_1 r_{\epsilon,1}\epsilon_0^{ab}\partial^b, \qquad (B.35)$$

$$E_{\phi t}R^a_{t\phi(0)} = \upsilon^\ominus r_{0,5}\partial^a - \upsilon^\ominus r_{0,5}\alpha^a\bar{\alpha}\partial - \upsilon^\ominus r_{0,5}\epsilon_0^{ab}\partial^b, \qquad (B.36)$$

we find

$$\widetilde{r}_{0,5} = 1, \qquad \widetilde{r}_{\epsilon,1} = -1, \qquad \widetilde{r}_{0,1} = 1, \qquad \widetilde{r}_{1,1} = -1. \qquad (B.37)$$

(v) Using the $\phi t$ part of equation (B.14) and the relation

$$E_{\phi\phi(2)}R^a_{\phi t(0)} + \left(E_{t\phi}R^a_{\phi\phi(1)}\right)^{\dagger} = (r_{0,4} - r_{1,1}\upsilon^\ominus)\bar{\alpha}\partial + (r_{0,4} - r_{\epsilon,1}\upsilon^\ominus)\epsilon_0^{ab}\partial^b\bar{\alpha}\partial, \qquad (B.38)$$

we obtain

$$\widetilde{r}_{0,4} = -1, \qquad \widetilde{r}_{0,4} = \widetilde{r}_{\epsilon,1}. \qquad (B.39)$$

(vi) Using $\widetilde{r}_{0,1}, \widetilde{\bar{r}}_{0,1}, \widetilde{r}_{0,4}, \widetilde{r}_{0,5}, \widetilde{r}_{1,1}, \widetilde{r}_{1,5}, \widetilde{r}_{\epsilon 1}$ given above, we make sure that all the remaining equations in (B.13)–(B.16) are satisfied automatically. We note that in the analysis of the $tt$ part of equation (B.15), we use the identity

$$\langle t|\alpha^a\bar{\alpha}\partial|t\rangle = \langle t|\alpha\partial\bar{\alpha}^a|t\rangle, \qquad (B.40)$$

which can be proved by using the self-duality constraint (5.10).

## Appendix C. On-shell D.o.F. of the self-dual field in 4$d$

We analyze the on-shell D.o.F. of the conformal self-dual field in 4$d$ with the Lagrangian (3.5). To this end we use the light-cone gauge. In the light-cone frame, the spacetime coordinates $x^a$ are decomposed as $x^a = x^+, x^-, x^i$, where the light-cone coordinates in $\pm$ directions are defined as $x^\pm = (x^3 \pm x^0)/\sqrt{2}$ and $x^+$ is taken to be a light-cone time. The $so(2)$ algebra vector indices take values $i, j = 1, 2$. We adopt the conventions $\partial^i = \partial_i \equiv \partial/\partial x^i$, $\partial^\pm = \partial_\mp \equiv \partial/\partial x^\mp$.

We are going to prove that the on-shell D.o.F. of the self-dual field are described by two $so(2)$ algebra self-dual complex-valued vector fields $\phi^i_{-1}, \phi^i_1$ and one complex-valued scalar field $\phi_0$:

$$\phi^i_{-1}, \qquad \phi^i_1, \qquad \phi_0, \qquad (C.1)$$

which satisfy the equations of motion

$$\Box\phi^i_{-1} - \phi^i_1 = 0\,, \qquad \Box\phi^i_1 = 0\,, \qquad \Box\phi_0 = 0\,. \tag{C.2}$$

The vector fields satisfy the $so(2)$ self-duality constraint:

$$\pi^{ij}\phi^j_{-1} = 0\,, \qquad \pi^{ij}\phi^j_1 = 0\,. \tag{C.3}$$

Here and below, we use the notation

$$\pi^{ij} \equiv \delta^{ij} + \tau\epsilon^{ij}\,, \qquad \bar\pi^{ij} \equiv \delta^{ij} - \tau\epsilon^{ij}\,, \tag{C.4}$$

where $\delta^{ij}$ is the Kronecker delta, while $\epsilon^{ij}$ is the Levi-Civita symbol normalized as $\epsilon^{12} = 1$.

In order to find on-shell D.o.F., we use equations of motion obtained from Lagrangian (3.5) and the self-duality constraint (3.2):

$$\partial^a F^{ab}(\phi_1) = 0\,, \tag{C.5}$$

$$\partial^b F^{ba}(\phi_{-1}) + \partial^b t^{ba} - \phi^a_1 - \partial^a\phi_0 = 0\,, \tag{C.6}$$

$$\Box\phi_0 + \partial^a\phi^a_1 = 0\,, \tag{C.7}$$

$$F^{ab}(\phi_1) = \frac{\tau}{2}\epsilon^{abce}F^{ce}(\phi_1)\,, \tag{C.8}$$

$$t^{ab} = \frac{\tau}{2}\epsilon^{abce}t^{ce}\,. \tag{C.9}$$

Taking into account the light-cone frame decomposition of the vector and tensor fields (3.1),

$$\phi^a_{-1} = \phi^+_{-1}, \phi^-_{-1}, \phi^i_{-1}, \qquad \phi^a_1 = \phi^+_1, \phi^-_1, \phi^i_1, \qquad t^{ab} = t^{+-}, t^{+i}, t^{-i}, t^{ij}, \tag{C.10}$$

we note that some of the gauge transformations given (3.10)–(3.13) can be represented as

$$\delta\phi^+_1 = \partial^+\xi_0\,, \tag{C.11}$$

$$\delta\phi^+_{-1} = \partial^+\xi_{-2} - \lambda^+\,, \tag{C.12}$$

$$\delta\phi^i_{-1} = \partial^i\xi_{-2} - \lambda^i\,, \tag{C.13}$$

$$\delta t^{+-} = \partial^+\lambda^- - \partial^-\lambda^+ + \tau\epsilon^{ij}\partial^i\lambda^j\,, \tag{C.14}$$

$$\delta t^{+i} = \bar\pi^{ij}(\partial^+\lambda^j - \partial^j\lambda^+)\,. \tag{C.15}$$

From (C.11), we see that the field $\phi^+_1$ can be gauge away by using the $\xi_0$ gauge transformation. From (C.12), (C.13), we see that the fields $\phi^+_{-1}$ and $\pi^{ij}\phi^j_{-1}$ can be gauged away by using the respective $\xi_{-2}$ and $\pi^{ij}\lambda^j$ gauge transformations. From (C.14), (C.15), we see that the fields $t^{+-}$ and $\bar\pi^{ij}t^{+j}$ can be gauged away by using the respective $\lambda^-$ and $\bar\pi^{ij}\lambda^j$ gauge transformations. To summarize, we can impose the following gauge conditions:

$$\phi^+_1 = 0\,, \tag{C.16}$$

$$\phi^+_{-1} = 0\,, \tag{C.17}$$

$$\pi^{ij}\phi^j_{-1} = 0\,, \tag{C.18}$$

$$\bar\pi^{ij}t^{+j} = 0\,, \tag{C.19}$$

$$t^{+-} = 0. \tag{C.20}$$

Using gauge conditions (C.16)–(C.20), one can make sure that equations (C.5)–(C.9) amount to the following equations:

$$\Box \phi_1^a = 0, \tag{C.21}$$

$$\partial^a \phi_1^a = 0, \tag{C.22}$$

$$\partial^a \phi_{-1}^a + \phi_0 = 0. \tag{C.23}$$

$$\Box \phi_{-1}^a + \partial^b t^{ba} - \phi_1^a = 0, \tag{C.24}$$

$$\Box \phi_0 = 0, \tag{C.25}$$

$$\pi^{ij} \phi_1^j = 0, \tag{C.26}$$

$$\pi^{ij} t^{+j} = 0, \tag{C.27}$$

$$\bar\pi^{ij} t^{-j} = 0, \tag{C.28}$$

$$t^{ij} = 0. \tag{C.29}$$

We now analyze gauge conditions (C.16)–(C.20) and equations (C.21)–(C.29).

(i) In view of (C.18), (C.26), we see that the $so(2)$ algebra vector fields $\phi_{-1}^i$, $\phi_1^i$ indeed satisfy the self-duality constraint given in (C.3).

(ii) Equation (C.21) leads to the second equation in (C.2).

(iii) Differential constraints (C.22), (C.23) and gauge conditions (C.16), (C.17) tell us that the non-dynamical fields $\phi_1^-$ and $\phi_{-1}^-$ can be expressed in terms of fields (C.1):

$$\phi_1^- = -\frac{\partial^i}{\partial^+} \phi_1^i, \qquad \phi_{-1}^- = -\frac{\partial^i}{\partial^+} \phi_{-1}^i - \frac{1}{\partial^+} \phi_0. \tag{C.30}$$

(iv) Equations (C.19), (C.27) imply

$$t^{+i} = 0. \tag{C.31}$$

(v) Taking into account (C.29), (C.31) and using equation (C.24), we obtain

$$\Box \phi_{-1}^i + \partial^+ t^{-i} - \phi_1^i = 0. \tag{C.32}$$

Multiplying equation (C.32) by $\bar\pi^{ij}$ and using constraint (C.18), we obtain the first equation in (C.2).

(vi) Multiplying equation (C.32) by $\pi^{ij}$ and taking into account (C.18), (C.26) gives the equation

$$\pi^{ij} t^{-j} = 0. \tag{C.33}$$

Equations (C.28), (C.33) imply

$$t^{-i} = 0. \tag{C.34}$$

Taking into account (C.20), (C.29), (C.31), (C.34), we see that $t^{ab} = 0$.

To summarize, we proved that on-shell D.o.F. of the self-dual field with Lagrangian (3.5) are described by fields given in (C.1). These fields satisfy equations of motion (C.2) and self-duality constraints (C.3). The light-cone gauge Lagrangian which leads to equations of motion (C.2) is given in (3.39).

## Appendix D. $\epsilon$-symbols

In this appendix, we describe various useful relations for $\epsilon$-symbols that we use in the paper. We introduce the following $\epsilon$-symbols constructed out of the Levi-Civita symbol $\epsilon^{a_1\ldots a_\nu b_1\ldots b_\nu}$ and the oscillators:

$$\epsilon \equiv \frac{\tau}{(\nu!)^2}\epsilon^{a_1\ldots a_\nu b_1\ldots b_\nu}\alpha^{a_1}\ldots\alpha^{a_\nu}\bar{\alpha}^{b_\nu}\ldots\bar{\alpha}^{b_1}, \tag{D.1}$$

$$\epsilon^a \equiv \frac{\tau}{\nu!(\nu-1)!}\epsilon^{a_1\ldots a_\nu ab_2\ldots b_\nu}\alpha^{a_1}\ldots\alpha^{a_\nu}\bar{\alpha}^{b_\nu}\ldots\bar{\alpha}^{b_2}, \tag{D.2}$$

$$\bar{\epsilon}^a \equiv \frac{\tau}{\nu!(\nu-1)!}\epsilon^{aa_2\ldots a_\nu b_1\ldots b_\nu}\alpha^{a_2}\ldots\alpha^{a_\nu}\bar{\alpha}^{b_\nu}\ldots\bar{\alpha}^{b_1}, \tag{D.3}$$

$$\epsilon_0^{ab} \equiv \frac{\tau}{((\nu-1)!)^2}\epsilon^{aa_2\ldots a_\nu bb_2\ldots b_\nu}\alpha^{a_2}\ldots\alpha^{a_\nu}\bar{\alpha}^{b_\nu}\ldots\bar{\alpha}^{b_2}, \tag{D.4}$$

$$\epsilon^{ab} \equiv \frac{\tau}{\nu!(\nu-2)!}\epsilon^{a_1\ldots a_\nu abb_3\ldots b_\nu}\alpha^{a_1}\ldots\alpha^{a_\nu}\bar{\alpha}^{b_\nu}\ldots\bar{\alpha}^{b_3}, \tag{D.5}$$

$$\bar{\epsilon}^{ab} \equiv \frac{\tau}{\nu!(\nu-2)!}\epsilon^{aba_3\ldots a_\nu b_1\ldots b_\nu}\alpha^{a_3}\ldots\alpha^{a_\nu}\bar{\alpha}^{b_\nu}\ldots\bar{\alpha}^{b_1}, \tag{D.6}$$

where $\tau$ is defined in (2.28). We note the following helpful relations for these $\epsilon$-symbols:

$$\epsilon^a \equiv [\epsilon,\alpha^a], \qquad \bar{\epsilon}^a \equiv [\bar{\alpha}^a,\epsilon], \tag{D.7}$$

$$\epsilon^{ab} \equiv \{[\epsilon,\alpha^a],\alpha^b\}, \qquad \bar{\epsilon}^{ab} \equiv \{\bar{\alpha}^a,[\bar{\alpha}^b,\epsilon]\}, \tag{D.8}$$

$$\epsilon_0^{ab} \equiv \{[\bar{\alpha}^a,\epsilon],\alpha^b\}, = \{\bar{\alpha}^a,\epsilon^b\} = \{\bar{\epsilon}^a,\alpha^b\}. \tag{D.9}$$

Our $\epsilon$-symbols satisfy the following Hermitian conjugation rules:

$$\epsilon^\dagger = -\epsilon, \qquad \epsilon^{a\dagger} = -\bar{\epsilon}^a, \qquad \epsilon_0^{ab\dagger} = \epsilon_0^{ab}, \qquad \epsilon^{ab\dagger} = \bar{\epsilon}^{ab}. \tag{D.10}$$

On the space of the ket-vector $|\phi\rangle$ subject to the constraint

$$N_\alpha|\phi\rangle = \nu|\phi\rangle, \tag{D.11}$$

we obtain the relation

$$\epsilon^2|\phi\rangle = |\phi\rangle. \tag{D.12}$$

It is this property of the $\epsilon$-symbol that is used for the definition of the self-dual ket-vector $|t\rangle$ (5.10). One has the following helpful identities involving $\epsilon$-symbols, the oscillators and the derivative:

$$\alpha\partial\epsilon_0^{ab}\partial^b = \epsilon^b\partial^b\partial^a - \epsilon^a\,\Box + \epsilon^{ab}\bar{\alpha}\partial\partial^b, \tag{D.13}$$

$$\epsilon_0^{ab}\bar{\alpha}\partial\partial^b = -\bar{\epsilon}^b\partial^b\partial^a + \bar{\epsilon}^a\,\Box + \alpha\partial\bar{\epsilon}^{ab}\partial^b. \tag{D.14}$$

## References

[1] Schwarz J H and West P C 1983 *Phys. Lett.* B **126** 301
[2] Marcus N and Schwarz J H 1982 *Phys. Lett.* B **115** 111
[3] Berkovits N 1997 *Phys. Lett.* B **395** 28 (arXiv:hep-th/9610134)
[4] Pasti P, Sorokin D P and Tonin M 1997 *Phys. Rev.* D **55** 6292 (arXiv:hep-th/9611100)
[5] Bandos I, Bekaert X, de Azcarraga J A, Sorokin D and Tsulaia M 2005 *J. High Energy Phys.* JHEP05(2005)031 (arXiv:hep-th/0501113)

[6]    Fradkin E S and Tseytlin A A 1985 *Phys. Rep.* **119** 233
[7]    Zinoviev Yu M 2001 On massive high spin particles in (A)dS arXiv:hep-th/0108192
[8]    Metsaev R R 2006 *Phys. Lett.* B **643** 205 (arXiv:hep-th/0609029)
[9]    Lopatin V E and Vasiliev M A 1988 *Mod. Phys. Lett.* A **3** 257
[10]   Vasiliev M A 1988 *Nucl. Phys.* B **301** 26
[11]   Labastida J M F 1989 *Nucl. Phys.* B **322** 185
[12]   Metsaev R R 1995 *Phys. Lett.* B **354** 78
[13]   Bekaert X and Boulanger N 2004 *Commun. Math. Phys.* **245** 27 (arXiv:hep-th/0208058)
[14]   Bekaert X and Boulanger N 2007 *Commun. Math. Phys.* **271** 723 (arXiv:hep-th/0606198)
[15]   Hallowell K and Waldron A 2005 *Nucl. Phys.* B **724** 453 (arXiv:hep-th/0505255)
[16]   Artsukevich A Y and Vasiliev M A 2009 *Phys. Rev.* D **79** 045007 (arXiv:0810.2065 [hep-th])
[17]   Engquist J and Sundell P 2006 *Nucl. Phys.* B **752** 206 (arXiv:hep-th/0508124)
[18]   Bastianelli F, Corradini O and Latini E 2007 *J. High Energy Phys.* JHEP02(2007)072 (arXiv:hep-th/0701055)
[19]   Bastianelli F, Corradini O and Latini E 2008 *J. High Energy Phys.* JHEP11(2008)054 (arXiv:0810.0188
        [hep-th])
[20]   Bergshoeff E, de Roo M and de Wit B 1981 *Nucl. Phys.* B **182** 173
[21]   Bergshoeff E, de Roo M and de Wit B 1983 *Nucl. Phys.* B **217** 489
[22]   Bergshoeff E, Sezgin E and Van Proeyen A 1986 *Nucl. Phys.* B **264** 653
        Bergshoeff E, Sezgin E and Van Proeyen A 2001 *Nucl. Phys.* B **598** 667
[23]   Bergshoeff E, Sezgin E and Van Proeyen A 1999 *Class. Quantum Grav.* **16** 3193 (arXiv:hep-th/9904085)
[24]   Bergshoeff E, Salam A and Sezgin E 1987 *Nucl. Phys.* B **279** 659
[25]   Metsaev R R 2007 Ordinary-derivative formulation of conformal low spin fields arXiv:0707.4437 [hep-th]
        Metsaev R R 2007 Ordinary-derivative formulation of conformal totally symmetric arbitrary spin bosonic fields
        arXiv:0709.4392 [hep-th]
[26]   Metsaev R R 2008 *Phys. Rev.* D **78** 106010 (arXiv:0805.3472 [hep-th])
[27]   Zinoviev Yu M 2007 *Nucl. Phys.* B **770** 83 (arXiv:hep-th/0609170)
[28]   Metsaev R R 2008 *Phys. Rev.* D **77** 025032 (arXiv:hep-th/0612279)
[29]   Siegel W 1999 *Fields* arXiv:hep-th/9912205
[30]   Fotopoulos A and Tsulaia M 2009 *Int. J. Mod. Phys.* A **24** 1 (arXiv:0805.1346 [hep-th])
[31]   Buchbinder I L, Krykhtin V A and Ryskina L L 2009 *Mod. Phys. Lett.* A **24** 401 (arXiv:0810.3467 [hep-th])
[32]   Zinoviev Yu M 2002 On massive mixed symmetry tensor fields in Minkowski space and (A)dS
        arXiv:hep-th/0211233
        Zinoviev Yu M 2003 First order formalism for mixed symmetry tensor fields arXiv:hep-th/0304067
        Zinoviev Yu M 2003 First order formalism for massive mixed symmetry tensor fields arXiv:hep-th/0306292
[33]   Skvortsov E D 2008 *J. High Energy Phys.* JHEP07(2008)004 (arXiv:0801.2268 [hep-th])
        Skvortsov E D 2009 *Nucl. Phys.* B **808** 569 (arXiv:0807.0903 [hep-th])
[34]   Alkalaev K B, Grigoriev M and Tipunin I Y 2009 *Nucl. Phys.* B **823** 509 (arXiv:0811.3999 [hep-th])
[35]   Moshin P Y and Reshetnyak A A 2007 *J. High Energy Phys.* JHEP10(2007)040 (arXiv:0707.0386 [hep-th])
[36]   Shaynkman O V, Tipunin I Y and Vasiliev M A 2006 *Rev. Math. Phys.* **18** 823 (arXiv:hep-th/0401086)
[37]   Vasiliev M A 2009 Bosonic conformal higher-spin fields of any symmetry arXiv:0909.5226 [hep-th]
[38]   Buchbinder I L and Galajinsky A V 2008 *J. High Energy Phys.* JHEP11(2008)081 (arXiv:0810.2852 [hep-th])
[39]   Campoleoni A, Francia D, Mourad J and Sagnotti A 2009 *Nucl. Phys.* B **815** 289 (arXiv:0810.4350 [hep-th])